A First Class Constraint Generates Not a Gauge Transformation, But a (Bad) Physical Change: The Cases of Maxwell and GR

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Abstract

In Dirac-Bergmann constrained dynamics, a first-class constraint typically does not *alone* generate a gauge transformation. By direct calculation it is found that each first-class constraint in Maxwell's theory generates a change in the electric field $\vec{E}$ by an arbitrary gradient, spoiling Gauss's law. The secondary first-class constraint $p^i,_i = 0$ still holds, but being a function of derivatives of momenta, it is not directly about $\vec{E}$ (a function of derivatives of A_μ). Only a special combination of the two first-class constraints, the Anderson-Bergmann (1951)-Castellani gauge generator G, leaves $\vec{E}$ unchanged. This problem is avoided if one uses a first-class constraint as the generator of a *canonical transformation*; but that partly strips the canonical coordinates of physical meaning as electromagnetic potentials and makes the electric field depend on the smearing function, bad behavior illustrating the wisdom of the Anderson-Bergmann (1951) Lagrangian orientation of interesting canonical transformations.

The need to keep gauge-invariant the relation $\dot{q} - \frac{\delta H}{\delta p} = -E_i - p^i = 0$ supports using the total Hamiltonian rather than the extended Hamiltonian. The results extend the Lagrangian-oriented reforms of Castellani, Sugano, Pons, Salisbury, Shepley, *etc.* by showing the inequivalence of the extended Hamiltonian to the total Hamiltonian (and hence the Lagrangian) even for *observables*, properly construed in the sense implying empirical equivalence.

Dirac and others have noticed the arbitrary velocities multiplying the primary constraints outside the canonical Hamiltonian while apparently overlooking the corresponding arbitrary coordinates multiplying the secondary constraints *inside* the canonical Hamiltonian, and so wrongly ascribed the gauge quality to the primaries alone, not the primary-secondary team G. Hence the Dirac conjecture about secondary first-class constraints rests upon a false presupposition. The usual concept of Dirac observables should also be modified to employ the gauge generator G, not the first-class constraints separately, so that the Hamiltonian observables become equivalent to the Lagrangian ones such as the electromagnetic field $F_{\mu\nu}$.

An appendix discusses analogous calculations for GR and sketches their conceptual consequences.

Keywords: Dirac-Bergmann constrained dynamics; gauge transformations; canonical quantization; observables; Hamiltonian methods

Contents

1 Introduction

In the early stages of research into constrained Hamiltonian dynamics by Bergmann's school, it was important to ensure that the new Hamiltonian formalism agreed with the established Lagrangian formalism. That was very reasonable, for what other criteria for success were there at that stage? One specific manifestation of Hamiltonian-Lagrangian equivalence was the recovery of the usual 4-dimensional Lagrangian gauge transformations for Maxwell's electromagnetism and (more laboriously) GR by Anderson and Bergmann [1]. 4-dimensional Lagrangian-equivalent gauge transformations were implemented by Anderson and Bergmann in the Hamiltonian formalism using the gauge generator (which I will call G), a specially tuned sum of the first-class constraints, primary and secondary, in electromagnetism or GR [1].

At some point, early on and explicitly in Dirac's work and increasingly in a tacit way by the mid-1950s among Bergmann and collaborators, equivalence with 4-dimensional Lagrangian considerations came to play a less significant role. Instead the idea that a first-class constraint *by itself* generates a gauge transformation became increasingly prominent. That claim, which goes back to Bergmann and Dirac [2–5], has been called the "'standard'" interpretation [6] and is adopted throughout Henneaux and Teitelboim's book [7, pp. 18, 54] and countless other places [8–10]. This idea displaced the Anderson-Bergmann gauge generator until the 1980s and remains a widely held view, though no longer a completely dominant one in the wake of the Lagrangian-oriented reforms of Castellani, Sugano, Pons, Salisbury, Shepley, *etc.* Closely paralleling the debate between the Lagrangian-equivalent gauge generator G and the distinctively Hamiltonian idea that a first-class constraint generates a gauge transformation is the debate between the Lagrangian-equivalent total Hamiltonian (which adds to the canonical Hamiltonian all the primary constraints, whether first- or second-class) and Dirac's extended Hamiltonian H_E, which adds to the total Hamiltonian the first-class secondary constraints.

A guiding theme of Pons, Shepley, and Salisbury's series of works [11–13] is important:

> We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent (see . . .) in coming to the conclusion that they in fact are. [14, p. 17; embedded reference is to [15]]

While proponents of the total Hamiltonian have emphasized the value of making the Hamiltonian formalism equivalent to the Lagrangian, what has apparently been lacking until now is a *proof* that the Lagrangian-inequivalent extended Hamiltonian is erroneous. While inequivalence of the extended Hamiltonian to the Lagrangian might seem worrisome, it is widely held that the difference is confined to gauge-dependent unobservable quantities and hence makes no real physical difference. If that claim of empirical equivalence were true, it would be a good defense of the permissibility of extending the Hamiltonian. But is that claim of empirical equivalence true?

This paper shows that the Lagrangian-equivalent view of the early Anderson-Bergmann work [1] and the more recent Lagrangian-oriented reforms are correct, that is, are mandatory rather than merely an interesting option. It does so by showing by direct calculation that a first-class constraint makes an observable difference to the observable electrical field, indeed a bad difference: it spoils Gauss's law $\nabla \cdot \vec{E} = 0$. The calculation is perhaps too easy to have seemed worth checking to most authors.

This paper also critiques the usual Hamiltonian-focused views of observables deployed in the extended Hamiltonian tradition to divert attention from such a calculation or (in the case of the one paper known to me that calculates the relevant Poisson brackets [16]) to explain away the embarrassment of a Gauss's law-violating change in the electric field. Attention is paid to which variables have physical meaning when (off-shell *vs.* on-shell), *etc.*, with the consequence that canonical momenta have observable significance only derivatively and on-shell rather than primordially and off-shell. The fact that introducing a Hamiltonian formalism neither increases nor decreases one's experimental powers is implemented consistently. Indeed apart from constraints, canonical momenta play basically the role of auxiliary fields in the Hamiltonian action $\int dt(p\dot{q} - H(q,p))$: one can vary with respect to p, get an equation $\dot{q} - \frac{\delta H}{\delta p} = 0$ to solve for p, and then use it to eliminate p from the action, getting $\int dtL$. One would scarcely call an auxiliary field a primordial observable and the remaining dependence on q or its derivatives in L derived.

This paper also diagnoses a mistaken 'proof' that a first-class primary constraint generates a gauge transformation. This mistake in Dirac's book [5] has been copied in various places, including two more recent books [7, 10]. One can see by inspection that the 3-vector potential A_i is left alone by the sum of first-class primary constraints, while the scalar potential is changed. But the science of electrostatics [17] explores the physical differences associated with different scalar potentials A_0 and the same (vanishing) 3-vector potential A_i. Thus Dirac *et al.* have pronounced observably different electric fields to be gauge-related. Dirac's mistake involves failing to note the term $-A_0 p^i,_i$ in the canonical Hamiltonian density for electromagnetism. Thinking that the secondary constraints either were absent or cancelled out in different evolutions (which they do not because the coefficient $-A_0$ of the secondary constraint is gauge-dependent), Dirac felt the need to add in the secondary first-class constraints by hand, extending the Hamiltonian, in order to recover the gauge freedom that supposedly was missing. Thus the motivation for the extended Hamiltonian and the original 'proof' that primary first-class constraints generate gauge transformations are dispelled.

This paper also explores the consequences for Dirac's conjecture that all first-class secondary constraints generate gauge transformations. That conjecture was predicated on the assumed validity of the proof that primary first-class constraints generate gauge transformations. With that proof refuted, the Dirac conjecture cannot even get started; it rests on a false presupposition.

The actual situation is quite the reverse of the idea that a first-class constraint generates a gauge transformation: the most obvious interesting examples of first-class constraints, as in Maxwell's electromagnetism and in General Relativity, *change the physical state or history*, and in a bad way, spoiling the Lagrangian constraints, the constraints in terms of q and $\dot{q}$. Those are the physically relevant constraints, parts of Maxwell's equations (Gauss's law) or the Einstein equations; the canonical momenta p are merely auxiliary quantities useful insofar as they lead back to the proper behavior for q and $\dot{q}$. While there might be examples where a first-class constraint does generate a gauge transformation—*e.g.*, —such cases are rare or uninteresting in comparison to those that do not.[1] Instead, a gauge transformation is generated by a *special combination* of first-class constraints, namely, the gauge generator G [1, 20–22]. It long was easy to neglect 4-dimensional coordinate transformations in GR because a usable gauge generator was unavailable after the $3+1$ split innovation in 1958 [23, 24] rendered the original (rather fearsome) G [1] obsolete by trivializing the primary constraints. The $3+1$ gauge generator G finally appeared in 1982 [20], the lengthy delay indicating that no one was looking for it for a long time.

For Maxwell's electromagnetism, where everyone knows what a gauge transformation is—what makes no physical difference, namely, leaving $\vec{E}$ and $\vec{B}$ unchanged—and where all the calculations are easy, one can *test* the claim that a first-class constraint generates a gauge transformation. There is no room for "interpretation," "definition," "assumption," "demand," or the like. Additional postulates are either redundant or erroneous. Surprisingly, given the age of the claim [2, 3], such a test apparently hasn't been made before, at least not completely and successfully (*c.f.* [25–28], on which more below), and has rarely been attempted. Perhaps the temptation to default to prior knowledge has been irresistible. By now the sanction of tradition and authority also operate. Views about observability have also deflected attention away from the question in the context of the extended Hamiltonian. Anyway the test can be made by re-mathematizing the verbal formula. The result is clearly negative: a first-class constraint—either the primary or the secondary—generates a physical difference, a change in $\vec{E}$. This change involves the gradient of an arbitrary function, implying that $\nabla \cdot \vec{E} \neq 0$, spoiling Gauss's law. Similar problems arise in GR, as will be discussed in a subsequent work in preparation. An error early in Dirac's book contributed to the problem; the same problem reappears in the books by Henneaux and Teitelboim and

[1] A free relativistic particle with all 4 coordinates as dynamical functions of an arbitrary parameter, but without an auxiliary lapse function N, is an example kindly mentioned by Josep Pons. If one has the auxiliary lapse function [11, 18], one gets a primary and a secondary constraint, the latter including a piece quadratic in momenta—looking naively like a Hamiltonian, one might say. If one instead integrates out the lapse using $\frac{\partial L}{\partial N} = 0$, then the resulting Hamiltonian formalism has vanishing canonical Hamiltonian, while the primary constraint becomes more interesting. Conserving the primary constraint gives no secondary or higher constraint, partly because the canonical Hamiltonian vanishes. The solitary primary constraint is first-class by antisymmetry of the Poisson bracket. In the absence of higher-order constraints, the gauge generator is just the smeared primary first-class constraint, so in this case a primary constraint does indeed generate a gauge transformation. A free relativistic particle is of course a system for which nothing happens. Potentially more interesting is the fact that one can integrate out the lapse in GR as in the Baierlein-Sharp-Wheeler action. Then the Hamiltonian constraint arises at the primary level [19].

by Rothe and Rothe [7, 10].

An alternative use of a first-class constraint, using it as a generating function in a canonical transformation, is also considered. While not illegal, such a canonical transformation is unrelated to electromagnetic gauge freedom (making as much sense for Proca's massive electromagnetism with only second-class constraints as for Maxwell's with only first-class constraints) and, as Anderson and Bergmann [1] would have predicted, alters the physical significance of the canonical field variables.

2 Expected Payoff of Clarity about First-Class Constraints and Gauge Transformations

While the process of Lagrangian-equivalent reform started some time ago, it has by no means swept the field. One also finds works that inconsistently mix the two views. While such issues cause little trouble in electromagnetism because all calculations are easy and one already knows all the right answers anyway and so does not depend on the Hamiltonian formalism, it does matter for GR, where the right answers are sometimes unknown or controversial and many calculations are difficult.

It is therefore important both to show that the extended Hamiltonian formalism and associated view of gauge freedom are incorrect (as this paper does) and to implement consistently the consequences of the Lagrangian-equivalent Hamiltonian formalism in the arenas of change and observables in GR (as successor papers will do). It has been widely held (or worried) that GR in Hamiltonian form lacks objective change [4, 29–32]. It also has been widely held in the Hamiltonian context, that "observables" in GR must be constants of the motion, spatially integrated quantities, or the like [33].

Both these conclusions are motivated largely by the alleged result that a first-class constraint generates a gauge transformation. Once one realizes that a first-class constraint by itself does not generate a gauge transformation, the fact that the Hamiltonian of GR is just a sum of first-class constraints no longer implies, or even suggests, that time evolution is just a gauge transformation. Instead room is left for showing that the Hamiltonian formalism discloses time-dependence in exactly the same context as the Lagrangian formalism, namely, when there is no time-like Killing vector field. Likewise one is relieved of the expectation that an observable quantity should have vanishing Poisson brackets with all of the first-class constraints; instead one might expect observables to have vanishing Poisson bracket with the gauge generator G. (Of course additional modification might be necessary for Lagrangian equivalence in relation to

GR, where the symmetry is external and one anticipates Lie derivative terms; but replacing the first-class constraints with G is still a step in the right direction.) While applications to GR will be saved for another work due to the amount of calculation involved, achieving clarity about first-class constraints and gauge transformations in Maxwell's electromagnetism will be a useful step.

3 A First-Class Primary Constraint Does Not Generate a Gauge Transformation

It is widely held [5, p. 21] [7, p. 17] [34] [10, p. 68] that a primary first-class constraint generates a gauge transformation. Dirac purportedly proves this claim early in his book, and the same argument reappears in many places including authoritative books, some of them not very old. In a later section the tempting error that leads to this conclusion, namely, neglecting the fact that first-class secondary constraints with gauge-dependent coefficients already appear in the canonical Hamiltonian, will be discussed. For now a direct and apparently novel (surprisingly enough) test will be applied to show simply *that* the transformation effected by a first-class primary constraint is not generally a gauge transformation. The test is simply ascertaining what happens to the electric field in Maxwell's electromagnetism, the standard example of a simple yet physically relevant relativistic field theory.

The electromagnetic field strength $F_{\mu\nu} =_{df} \partial_\mu A_\nu - \partial_\nu A_\mu$ is unchanged by $A_\mu \rightarrow A_\mu - \partial_\mu \epsilon$. $\vec{E}$ and $\vec{B}$ are parts of $F_{\mu\nu}$ and hence constructed from derivatives of A_μ. (For a charged particle in an electromagnetic field, or for a charged scalar field interacting with the electromagnetic field, it is the derivatives of A_μ, not the canonical momentum conjugate to A_μ, to which charge responds.) That fact will prove important once, in the Hamiltonian formulation, one has conceptually independent canonical momenta p^i satisfying the secondary first-class constraint $p^i,_i = 0$. Electromagnetic gauge transformations are defined "off-shell," without assuming the field equations. But off-shell there is no relationship between $\dot{A}_i$ and p^i, and hence none between $\vec{E}$ and p^i. The constraint $p^i,_i = 0$ in phase space can cease to be equivalent to $\nabla \cdot \vec{E} = 0$ if one does something inadvisable—such as treating p^0 or $p^i,_i$ as if it (by itself) generated a gauge transformation. That is somewhat as Anderson and Bergmann warned in discussing canonical transformations that do not reflect Lagrangian invariances: the meanings of the canonical coordinates and/or momenta can be changed [1]. The relationship between first-class constraints, the gauge generator G, and canonical transformations will be explored below. It turns out that G does basically the same good thing whether one simply takes Poisson brackets directly or makes a canonical transformation; a first-class constraint does either something permitted but pointless (a position-dependent field redefinition) or something disastrous (spoiling Gauss's law).

The Legendre transformation from $\mathcal{L}$ and $\dot{A}_\mu$ to $\mathcal{H}$ and p^μ fails because $p^\mu =_{df} \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$ is not soluble for $\dot{A}_\mu$ [35]. One gets a primary constraint $p^0(x) =_{df} \frac{\partial \mathcal{L}}{\partial A_{0,0}} = 0$. Likewise in General Relativity [23, 24], one can choose a divergence in $\mathcal{L}$ and a set of fields using a $3+1$ split, the lapse $N = 1/\sqrt{-g^{00}}$ and shift vector $\beta^i = {}^3g^{ij} g_{j0}$, such that $p_0 =_{df} \frac{\partial \mathcal{L}}{\partial N_{,0}} = 0$ and $p_i =_{df} \frac{\partial \mathcal{L}}{\partial N^i{}_{,0}} = 0$. One needs the dynamical preservation of the primary constraints, from which emerge secondary constraints. In electromagnetism this constraint is Gauss's law, or rather, something equivalent to Gauss's law using $\dot{A}_i = \frac{\delta H}{\delta p^i}$. The algorithm of constraint preservation terminates thanks to the constraint algebra. The time evolution is under-specified: there is gauge/coordinate freedom due to the presence of first-class constraints (having 0 Poisson brackets among themselves, strongly in electromagnetism, using the constraints themselves in GR). All constraints in both theories are first-class. The Poisson bracket is

$$\{\phi(x), \psi(y)\} =_{df} \int d^3z \sum_A \left(\frac{\delta \phi(x)}{\delta q^A(z)} \frac{\delta \psi(y)}{\delta p_A(z)} - \frac{\delta \phi(x)}{\delta p_A(z)} \frac{\delta \psi(y)}{\delta q_A(z)} \right);$$

the fundamental ones are $\{q^A(x), p_B(y)\} = \delta^A_B \delta(x, y)$.

These familiar matters set up the belated *test* of whether a first-class constraint really generates a gauge transformation. *Exactly what* do first-class constraints have to do with gauge freedom? Curiously, this question has two standard but incompatible answers in the literature on constrained dynamics, both dating to the 1950s in Bergmann's work. One of them is correct, namely, that the gauge generator G [1, 20–22] generates a gauge transformation, a change in the description of the physical state (or history, if GR is the theory in question) that makes no objective difference. This answer is motivated by Hamiltonian-Lagrangian equivalence and is associated with the total Hamilton. It was eclipsed during the 1950s and has slowly reappeared since the 1980s. However, its consequences for observables, change in GR, and similar foundational questions have not been fully explored yet. The other standard answer, more influential in the literature on canonical GR, is that a first-class constraint (by itself) generates a gauge transformation [2, 3, 6, 7, 10, 25, 34], a distinctively Hamiltonian claim associated with the extended Hamiltonian.

In electromagnetism the fundamental Poisson brackets are $\{A_\mu(x), p^\nu(y)\} = \delta^\nu_\mu \delta(x, y)$. The constraints are the primary $p^0(x) = 0$ and the secondary $p^i{}_{,i}(x) = 0$.

One hopes to keep the latter equivalent to Gauss's law, but that isn't just automatic because Gauss's law involves the electric field, whereas the secondary constraint involves a canonical momentum, which *a priori* is unrelated to the electric field and becomes equal to it (up to a sign, depending on one's conventions) only using the equations of motion $\dot{q} = \frac{\delta H}{\delta p}$.

What does $p^0(x)$ do? By re-mathematizing the claim that a first-class constraint generates a gauge transformation, one predicts that $p^0(x)$ changes A_μ *via* a gauge transformation. Smearing $p^0(y)$ with arbitrary $\xi(t,y)$ and taking the Poisson bracket gives [35, p. 134]

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y p^0(y)\xi(t,y)\} = \delta^0_\mu \xi(t,x). \tag{1}$$

While this expression doesn't look just as one would expect from experience with the Lagrangian, might it reflect (as is oftened claimed abstractly) some more general gauge invariance disclosed by the Hamiltonian (especially the extended Hamiltonian) formalism? One can calculate that

$$\delta F_{\mu\nu} =_{df} F_{\mu\nu}[A + \delta A] - F_{\mu\nu}[A] = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \partial_\mu \xi \delta^0_\nu - \partial_\nu \xi \delta^0_\mu. \tag{2}$$

This definition reflects the standard gauge variation of a velocity as the time derivative of the gauge variation of the corresponding coordinate. Letting $\mu = m$, $\nu = n$, one sees that the magnetic field is invariant [35, p. 134], which is a good sign.

What happens to the electric field $\vec{E}$? Here Sundermeyer stops short [35, p. 134].[2] Let $\mu = 0$, $\nu = n$:

$$\delta F_{0n} = -\delta\vec{E} = \partial_0 \delta A_n - \partial_n \delta A_0 = \partial_0 \xi \delta^0_n - \partial_n \xi \delta^0_0 = -\partial_n \xi. \tag{3}$$

Unless one restricts oneself to the very uninteresting special case of spatially constant ξ (perhaps still depending on time), this is not a gauge transformation, because the world is different, indeed worse.[3] While $\vec{B}$ is unchanged, $\vec{E}$ is changed by $\partial_n \xi(t,x)$. Thus Gauss's law $\nabla \cdot \vec{E} = 0$ is spoiled: $\nabla \cdot \vec{E} = \nabla^2 \xi \neq 0$ typically. This spoilage of the Lagrangian constraint is not immediately obvious because the secondary constraint $p^i{}_{,i} = 0$ still holds. The trouble is that this expression, which lives in phase space,

[2] Costa *et al.* [16] got the same mathematical result. They failed to discern that it was problematic physically, for reasons discussed below involving which fields are observable.

[3] This result shows the inadequacy of the view, which one sometimes hears, that a first-class constraint generates a time-independent gauge transformation. Even a time-independent $\xi(x)$ changes $\vec{E}$ and spoils Gauss's law.

ceases to mean what one expected. p is independent of q, but $\dot{q}$ is dependent on q by definition; hence $\dot{q}$ and p are independent, at least until *after* Poisson brackets are calculated. $\vec{E}$ is a familiar function of derivatives of A_μ; the change in A_μ implies a Gauss's law-violating change in $\vec{E}$. While still $p^i{}_{,i} = 0$ (the phase space constraint surface is preserved), this constraint is no longer equivalent to Gauss's law: $p^i{}_{,i} = 0$ but $\nabla \cdot \vec{E} \neq 0$. Instead $\vec{E}$ acts as though some phantom charge density were a source. The relationship between p and $\dot{q}$ has been altered, something that Anderson and Bergmann warned could happen [1]. Changing $\vec{E}$ is a physical difference, not a gauge transformation—indeed a bad physical difference, because spoiling Gauss's law is bad.

If a first-class constraint does not generate a gauge transformation, one might hope that a book on constrained dynamics would point that fact out. That expectation is almost fulfilled. Sundermeyer commented on the "vague relation between first class constraint transformations and local gauge transformations." [35, p. 134]. That was true, but an understatement. Sundermeyer appeared to be in the process of reinventing the gauge generator in the chapters on electromagnetism and Yang-Mills theories [35, pp. 134, 168], but did not carry on quite far enough to notice that something bad had happened to the electric field even after calculating what happened to the potentials. Thus he did not notice that the indirect relationship between first class constraint transformations and local gauge transformations that he discerned implied something crucially wrong with the usual view of the former.

3.1 Claims Overlooking This Problem

Others have fallen into error on this point [25, 27]. Bergvelt and de Kerg, applying their Hamiltonian technique to a Yang-Mills field,

> . . . first note that two points of [final constraint manifold] M_2 of the form (A_0, A, π) and $(\hat{A}_0, A, \pi)$ (i.e. differing only in their A_0-component) are gauge equivalent. They can be connected by an integral curve of the gauge vector field $\dot{A}(\frac{\delta}{\delta A_0})$, with $\dot{A} = \hat{A}_0 - A_0$. So the A_0-component of points of M_2 is physically irrelevant and without loss of generality we can ignore it. [27, p. 133].

This physical equivalence claim contradicts the science of electrostatics, wherein one studies what electric fields can be generated by merely the scalar potential [17, 36]. Presumably their "crucial *assumption*" that some freedom located in their preceding paper had no physical significance [37] contributed to this difficulty. One already knows from the Lagrangian formulation what the gauge freedom is, so there is no room for independent postulates; they are either redundant or erroneous. Gotay, Nester and Hinds make a similar mistake with the primary constraint [25], as will appear shortly.

4 A First-Class Secondary Constraint Does Not Generate a Gauge Transformation

What does the secondary constraint $p^i{}_{,i}(x)$ do? According to a standard textbook on constrained dynamics by Henneaux and Teitelboim, excepting a few exotic counterexamples,

> *one postulates, in general, that all first-class constraints generate gauge transformations.* This is the point of view adopted throughout this book. There are a number of good reasons to do this. First, the distinction between primary and secondary constraints, being based on the Lagrangian, is not a natural one from the Hamiltonian point of view.... Second, the scheme is consistent.... Third, as we shall see later, the known quantization methods for constrained systems put all first-class constraints on the same footing, *i.e.*, treat all of them as gauge generators. It is actually not clear if one can at all quantize otherwise. Anyway, since the conjecture holds in all physical applications known so far, the issue is somewhat academic. (A proof of the Dirac conjecture under simplifying regularity conditions that are generically fulfilled is given in subsection 3.3.2.) [7, p. 18, emphasis in the original]

This is a striking passage in view of the test that is about to be run on electromagnetism regarding its secondary constraint and the one that was just run above on the primary constraint. Getting sensible results does require privileging the Lagrangian formalism, so one should not downplay the primary *vs.* secondary distinction on Hamiltonian grounds. It would be interesting, but will not be attempted here, to trace all the influence of the Dirac conjecture in this standard work, as well as to address the third consideration about quantization methods (about which see [38]).

Another way to find out what the secondary constraint $p^i{}_{,i}$ does to the electric field is simply to *calculate it.* To my knowledge, this has not been done, surprisingly enough, or at least not done successfully and then appropriately understood. (Proponents of the total Hamiltonian and its gauge generator don't need to calculate it, because the usual gauge transformation of A_μ to $A_\mu - \partial_\mu \epsilon$ makes the answer obvious. Only proponents of the extended Hamiltonian and/or the associated claim that a first-class constraint generates a gauge transformation ought to have done so. But if they had, they'd likely have seen this problem before. Costa *et al.* did perform relevant calculations on this point [16]; the reason that they did not discern the absurdity of the result involves observables and will be discussed below.) The answer is the secondary first-class constraint also changes $\vec{E}$, also generally violating Gauss's law, at least if one uses a time-dependent smearing function. If one does not use time-dependent smearing functions, then one has no way to write G and hence no hope of recovering the usual electromagnetic gauge transformations as described in, for example, Jackson [17]. Part of the trouble, as diagnosed by Pons [39], is that Dirac envisioned gauge transformations as pertaining to 3-dimensional hypersurfaces, whereas Bergmann tended to envision them (more appropriately for GR given the freedom to slice more or less arbitrarily) as pertaining to 4-dimensional histories (though Bergmann seems to me not consistent on that point). Smearing $p^i{}_{,i}$ with an arbitrary function $\epsilon(t, y)$, one finds [16, 34]

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y p^i{}_{,i}(y)\epsilon(t, y)\} = -\delta^i_\mu \frac{\partial}{\partial x^i}\epsilon(t, x). \tag{4}$$

One can thus find the change in $F_{\mu\nu}$:

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \partial_\mu(-\delta^i_\nu \frac{\partial}{\partial x^i}\epsilon) - \partial_\nu(-\delta^i_\mu \frac{\partial}{\partial x^i}\epsilon) = \delta^i_\mu \partial_\nu \partial_i \epsilon - \delta^i_\nu \partial_\mu \partial_i \epsilon. \tag{5}$$

Clearly $\vec{B}$ is unchanged, but $\vec{E}$'s change is obtained by setting $\mu = 0$, $\nu = n$:

$$\delta F_{0n} = -\delta \vec{E} = \delta^i_0 \partial_n \partial_i \epsilon - \delta^i_n \partial_0 \partial_i \epsilon = -\partial_n \partial_0 \epsilon. \tag{6}$$

Again $\vec{E}$ is changed by an arbitrary gradient, and Gauss's law is spoiled: $\nabla \cdot \vec{E} = \nabla^2 \dot{\epsilon}$. One could avoid this change in $\vec{E}$ using exclusively time-independent smearing functions; but one will thereby fail to recover the usual electromagnetic gauge transformations in works like Jackson [17]. Imposing time-independence (or spatial homogeneity) on smearing functions is of course also incompatible with Lorentz invariance (to say nothing of general covariance for the analogous issue in GR).

So neither constraint by itself generates a gauge transformation (without a pointless and misleading restriction on smearing, at any rate, which restricts what the constraint itself is trying to generate). Each makes a bad physical difference. Dirac wrote that "I haven't found any example for which there exists first-class secondary constraints which do generate a change in the physical state." [5, p. 24] This remark now looks curious; it's not easy to find anything interesting that isn't a counterexample when the appropriate test is run. 30 years ago Castellani said that

> Dirac's conjecture that all secondary first-class constraints generate symmetries is revisited and replaced by a theorem.... The old question whether

secondary first-class constraints generate gauge symmetries or not ... is then solved: they are *part* of a gauge generator G ... [20, pp. 357, 358]. (emphasis in the original)

After many years the force of the word "replaced" still has not been absorbed (*e.g.*, [7]): it involves the *elimination of the old erroneous claim*, not just the introduction of a new true claim. Perhaps Castellani's diplomatic wording has slowed the understanding of his result. His target was the secondaries in isolation (supposedly the live issue *vis-a-vis* the Dirac conjecture), but the same holds for the primaries. Neither generates a gauge transformation by itself, but the two together, properly tuned, do.

4.1 Claims Overlooking This Problem

One can find examples where these problems should have been noticed. One is the influential paper by Gotay, Nester and Hinds [25]. (According to Web of Science, this paper has been cited *c.* 150 times.) Having developed a sophisticated theory, they rightly turned to applying it to Maxwell's electromagnetism. Having written the Hamiltonian field equations, they made a transverse-longitudinal split of the 3-vector potential $\vec{A}$ and its canonical momentum. They obtain, among other familiar results,

$$\frac{\partial A_\perp}{\partial t} = undetermined,$$

$$\frac{\partial \vec{A}_L}{\partial t} = -\nabla A_\perp.$$

Thus "the evolution of $A_\perp$ and $\vec{A}_\perp$ is arbitrary." [25] So far, so good—at least if one counts a *single* bit of arbitrariness, given that the arbitrariness in $-\nabla A_\perp$ determines the arbitrariness in the evolution of $\vec{A}_L$. Time will tell if that interpretation is maintained.

> Let us compare the equations of motion [of which the relevant parts just appeared] and the known gauge freedom of the electromagnetic field with the predictions of the algorithm.... [Something pertaining to the primary constraint has as] its effect to generate arbitrary changes in the evolution of $A_\perp$. This is clearly consistent with the field equations.

Well, it is consistent with the field equations if one pays the price by adding a gradient in $\frac{\partial \vec{A}_L}{\partial t}$ in accord with the familiar electromagnetic gauge freedom. But that turns out not to be what they have in mind.

> Turning now to the first-class secondary constraint ..., we wonder if it is the generator of physically irrelevant motions. ... [Imposing a suitable *demand*]

has the effect of replacing the second of equations [shown above] by

$$\frac{d\vec{A}_L}{dt} = -\vec{\nabla} A_\perp - \vec{\nabla} g$$

and leaving the others invariant. As $A_\perp$ is arbitrary to begin with, it is evident that this equation is completely equivalent to [the ones shown]. The addition of $-\vec{\nabla} g$ to the right-hand side of this equation has no physical effect whatsoever. [25, p. 2397].

It is now clear that they envisage two arbitrary functions, not one. But this latter physical equivalence claim is clearly false. Now that the former claim is disambiguated, it becomes clearly false also. Thus they wrongly claim of the primary and of the secondary that a gauge transformation is generated. By taking the divergence of the modified equation, one sees the falsehood of the second physical equivalence claim:

$$\begin{aligned} \vec{\nabla} \cdot \frac{\partial \vec{A}_L}{\partial t} + \vec{\nabla} \cdot \vec{\nabla} A_\perp + \vec{\nabla} \cdot \vec{\nabla} g = 0 \\ = \vec{\nabla} \cdot \frac{\partial (\vec{A}_L + \vec{A}_T)}{\partial t} + \vec{\nabla} \cdot \vec{\nabla} A_\perp + \vec{\nabla} \cdot \vec{\nabla} g \\ = \vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \cdot \vec{\nabla} A_\perp + \vec{\nabla} \cdot \vec{\nabla} g \\ = \vec{\nabla} \cdot (\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} A_\perp) + \vec{\nabla} \cdot \vec{\nabla} g = \\ \vec{\nabla} \cdot \vec{E} + \nabla^2 g = 0. \end{aligned} \tag{7}$$

Gotay, Nester and Hinds see their result as a vindication of the extended Hamiltonian formalism for the case of electromagnetism, but it isn't, because the electric field is changed by a so-called gauge transformation and Gauss's Law is spoiled. This problem illustrates a remark of Henneaux and Teitelboim's:

> The identification of the gauge orbits with the null surfaces of the induced two-form relies strongly on the postulate made throughout the book that all first-class constraints generate gauge transformations. If this were not the case, the gauge orbits would be strictly smaller than the null surfaces, and there would be null directions not associated with any gauge transformation. [7, p. 54]

Another difficulty appears in Faddeev's treatment [26], which, largely through notational confusion, gives the impression of showing that the constraint $p^i{}_{,i}$ generates a standard electromagnetic gauge transformation. He uses the symbol E_k for the canonical momentum conjugate to A_k. (Faddeev does not bother introducing a canonical momentum conjugate to A_0, so this paragraph will avoid the term "secondary constraint.") It isn't difficult to show that the canonical momentum E_k has vanishing Poisson bracket with the smeared constraint $\int d^3x \Lambda(x) \partial_k E_k$ for smearing function $\Lambda(x)$. But this result is hardly decisive for *the electric field*. Using the letter E for a canonical momentum cannot make a canonical momentum into the electric field, which is still just the familiar $A_{0},_i - \dot{A}_i$, which pushes on charged matter. Taking results about the canonical momentum and treating them as applying to the electric field is, in effect, the fallacy of equivocation regarding the meaning of E_k. Faddeev does not investigate, directly or indirectly, what a Poisson bracket with $\int d^3x \Lambda(x) \partial_k E_k$ does to $A_{0},_i - \dot{A}_i$. Hence the supposed demonstration that $\int d^3x \Lambda(x) \partial_k E_k$ generates an electromagnetic gauge transformation, fails. The relation between the electric field and the canonical momentum in facts holds only *on-shell*, that is, after all Poisson brackets are taken, because it reappears in the equation $\dot{q} = \frac{\delta H}{\delta p}$ after being discarded in the Legendre transformation. Hence showing that the canonical momentum has vanishing Poisson bracket with $\int d^3 \Lambda(x) \partial_k E_k$ does not show the same result for the electric field. If one hasn't defined a Poisson bracket for a velocity, one can at least ascertain what the smeared divergence of the canonical 3-momentum does to $A_{0},_i$ and A_i and then infer the altered $F_{\mu\nu}$ (as was just done above). If one defines a Poisson bracket for a velocity (following Anderson and Bergmann [1]), one can calculate the Poisson bracket of the electric field with the smeared divergence of the canonical 3-momentum and find that it isn't 0 (as is done below). Thus the smeared divergence of the canonical 3-momentum does not generate a gauge transformation. But the error seems to be tempting and to pass by without remark.

5 Gauge Generator as Special Sum of First-Class Constraints

While Dirac studies electromagnetism [5], his process of adding terms to and subtracting terms from the Hamiltonian is not systematic. Neither is there much concern to preserve equivalence to the Lagrangian formalism [40]. He seems not to calculate what his first-class constraints actually do.

One can add the two independently smeared constraints' actions together:

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y [p^0(y)\xi(t,y) + p^i,_i(y)\epsilon(t,y)]\} = \delta^0_\mu \xi - \delta^i_\mu \partial_i \epsilon, \tag{8}$$

getting their combined change in $\vec{E}$:

$$\delta F_{0n} = -\delta \vec{E} = -\partial_n \xi - \partial_n \partial_0 \epsilon. \tag{9}$$

If one puts the constraints to work together as a team by setting $\xi = -\dot{\epsilon}$ to make the $\delta F_{0n} = 0$, then

$$\delta A_\mu(x) = \{A_\mu(x), \int d^3y[-p^0(y)\dot{\epsilon}(t,y) + p^i{}_{,i}(y)\epsilon(t,y)]\} = -\delta^0_\mu \dot{\epsilon} - \delta^i_\mu \partial_i \epsilon = -\partial_\mu \epsilon, \tag{10}$$

which is good. Not surprisingly in light of the form of the gauge generator [1, 20, 22]

$$G = \int d^3x (p^i{}_{,i}\, \epsilon - p^0 \dot{\epsilon}), \tag{11}$$

p^0 and $p^i{}_{,i}$ generate *compensating* changes in $\vec{E}$ when suitably combined. Indeed we have pieced together G by demanding that the changes in $\vec{E}$ cancel out. Two wrongs, with opposite signs and time differentiation, make a right. This tuning, not surprisingly, is a special case of what Sundermeyer found necessary to get first-class transformations to combine suitably to get the familiar gauge transformation for the potentials for Yang-Mills [35, p. 168]. Sundermeyer, however, did not calculate the field strength(s) and notice the disastrous spoilage of the Gauss's law-type constraints by first-class transformations. Hence recovering the familiar gauge transformation of the potentials for him was merely a good idea.

One could make similar remarks about Wipf's treatment of Yang-Mills fields [34, p. 38], except that Wipf doesn't even seem to find recovering the Lagrangian gauge transformations a good idea; it's simply an option. If one doesn't have that taste, one at any rate has "the canonical symmetries" from an arbitrary sum of the first-class constraints [34, p. 37]; Wipf advocates extending the Hamiltonian [34, p. 28]. But what one actually one gets from an arbitrary sum of first-class constraints is the spoilage of Gauss's law. Combining the constraints to form the gauge generator is not an option (as in Wipf), nor even a good idea (as for Sundermeyer); it is compulsory. To my knowledge even the proponents of the gauge generator G and the total Hamiltonian have never shown that the extended Hamiltonian and its associated first-class-constraint-generates-a-gauge-transformation claim are *disastrous*.

Now with the primary and secondary constraints working together, Gauss's law is preserved: $\nabla \cdot \vec{E} = \nabla^2 \xi + \nabla^2 \dot{\epsilon} = \nabla^2(-\dot{\epsilon} + \dot{\epsilon}) = 0$. A first-class constraint typically does *not* generate a gauge transformation; it is *part* of the gauge generator G, which here acts as $\{A_\mu, G\} = -\partial_\mu \epsilon$, $\{p^\mu, G\} = 0$. Hence electromagnetism is just what Jackson says it is [17]; if a first-class constraint alone generated a gauge transformation, the Hamiltonian formulation would not be equivalent to the Lagrangian formulation.

Advocates of the gauge generator G combining the constraints [1, 20, 22] generally have aimed to recover the usual transformation of the *potential(s)* A_μ; the transformation of the field strength(s) $F_{\mu\nu}$ would follow obviously in the usual way and so did not need explicit calculation. Part of the contribution made here is to calculate the effects of a first-class constraint on the *field strength* $F_{\mu\nu}$, because calculating the effect on the gauge-invariant observable field strength leaves nowhere for error to hide. By taking the curl before tuning the sum of first-class constraints rather than after, one sees more vividly why that tuned sum is required and the separate pieces are unacceptable; one sees the looming disaster to be avoided, rather than avoiding it without seeing it. Beholding the resulting disaster makes the package involving the gauge generator G, the total Hamiltonian, and Lagrangian-equivalence *compulsory* in a way it previously has not seemed. The commutative diagram illustrates what differs and what is the same in commuting the operations of inferring $F_{\mu\nu}$ from A_μ and in inferring from effects of the tuned combination G from the effects of the separate first-class constraints:

$$\begin{array}{ccccccc}
A_\mu & \xrightarrow{L-equiv.} & G = \int d^3x(-p^0\dot{\epsilon} + \epsilon p^i{,_i}) & \longrightarrow & \delta A_\mu = -\partial_\mu \epsilon \\
{\scriptstyle \int d^3x(p^0\xi+\epsilon p^i,_i)}\downarrow & & & & \downarrow{\scriptstyle curl} \\
\delta A_\mu = \delta^0_\mu \xi - \delta^i_\mu \epsilon,_i & \xrightarrow{curl} & \delta F_{\mu\nu} = (\delta^0_\nu \xi,_\mu - \delta^i_\nu \epsilon,_{i\mu}) - \mu \leftrightarrow \nu & \xrightarrow[\xi=-\dot{\epsilon}]{L-equiv.} & \delta F_{\mu\nu} = 0
\end{array}$$

While the top line is fairly familiar, the bottom line appears to be novel, with the merely partial exception of ([16]). It is of course unacceptable to have $\delta F_{\mu\nu} \neq 0$, so requiring Lagrangian equivalence from the Hamiltonian resolves the trouble.

6 Gauge Invariance of $\dot{q} - \frac{\delta H}{\delta p} = -E_i - p^i = 0$

In the Lagrangian formalism, one *defines* the canonical momenta as $p_i =_{df} \frac{\partial L}{\partial q^i{,_0}}$. In that context, there is no difference in gauge transformation properties between $p_i =_{df} \frac{\partial L}{\partial q^i{,_0}}$; p_i simply inherits its gauge transformation behavior through this definition.

In the Hamiltonian formalism, one thing changes and another one doesn't. What changes is the gauge transformation behavior of p_i. In the Hamiltonian formalism p_i is independent, so it no longer inherits gauge transformation behavior from $\frac{\partial L}{\partial q^i{,_0}}$. Instead p_i gets its gauge transformation behavior somehow or other (together or separately) from Poisson brackets with first-class constraints. What does not change is the gauge transformation behavior of $\dot{q}^i$ (which in many examples is heavily involved in the Lagrangian gauge transformation behavior of $\frac{\partial L}{\partial q^i{,_0}}$).

One hopes, of course, to recover from the new Hamilton's equation $\dot{q}^i - \frac{\delta H}{\delta p_i} = 0$ what one had in the Lagrangian formalism in $p_i =_{df} \frac{\partial L}{\partial q^i{,_0}}$ and then gave up in setting the conjugate momenta free. On the other hand, if one is careless about gauge transformation properties of p_i or (more commonly) $\dot{q}^i$ in the Hamiltonian formalism, it is possible to spoil $\dot{q}^i - \frac{\delta H}{\delta p_i} = 0$. The equation $\dot{q}^i - \frac{\delta H}{\delta p_i} = 0$ holds only on-shell; it is not an identity in the Hamiltonian formalism. Thus one thing that one must not do (though one sometimes sees it done) is to pretend that one can use this equation to define the gauge transformation properties of $\dot{q}^i$. One cannot do that, because gauge transformations are generated using Poisson brackets, *i.e.*, off-shell, at the same logical 'moment' as the equations $\dot{q}^i = \frac{\delta H}{\delta p_i}$, which are also generated using Poisson brackets. Thus there is no relationship between $\dot{q}^i$ and $\frac{\delta H}{\delta p_i}$ at that stage. For the case of electromagnetism, there is no relationship between the electric field $\vec{E}$ (which is not quite $\dot{A}_i$, but is close enough) and the canonical momentum p^i (which is not quite $\frac{\delta H}{\delta p^i}$, but, again, is close enough). On the other hand, one *still knows* the gauge transformation behavior of the velocity $\dot{q}^i$, namely, the time derivative of the gauge transformation of q^i: $\delta \dot{q}^i = (\delta q)^i{,_0}$. For electromagnetism, this means roughly that one can simply calculate how the new $F_{\mu\nu}$ following from the new A_μ by the usual definition (taking the curl), differs from the old $F_{\mu\nu}$ derived from the old A_μ. The on-shell equality of $\dot{q}^i$ and $\frac{\delta H}{\delta p_i}$ thus imposes a condition of *on-shell equality of the gauge transformations* of $\dot{q}^i$ and $\frac{\delta H}{\delta p_i}$. This condition restricts what sorts of transformations can be gauge transformations. In the case at hand, $\vec{E}$ is roughly $\dot{A}_i$ (corrected by some unproblematic spatial derivatives of A_μ) and p^i is roughly $\frac{\delta H}{\delta p_i}$ (again, corrected by some unproblematic spatial derivatives of A_μ). Thus the condition is that the gauge-transformation properties of $\vec{E}$ and p^i agree on-shell. While p^i has vanishing Poisson bracket with each first-class constraint separately in this case, $\vec{E}$ has vanishing Poisson bracket only with the gauge generator G that combines the two first-class constraints so as to cancel out the change that each one makes separately. Gauge invariance of $\dot{q}^i = \frac{\delta H}{\delta p_i}$ thus necessitates regarding G as the gauge generator, and not regarding each isolated first-class constraint as generating a gauge transformation. That way, and only that way, one keeps $\dot{q}^i - \frac{\delta H}{\delta p_i} = 0$ gauge invariant. Otherwise it isn't clear what the rules of the Hamiltonian formalism are.

For the specific case of electromagnetism, one has the (canonical) Hamiltonian [35,

p. 127]

$$\int d^3x[\frac{1}{2}(p^i)^2 + \frac{1}{4}F_{ij}^2 - A_0 p^i{}_{,i}]. \tag{12}$$

Thus $\dot{q} - \frac{\delta H}{\delta p} = 0$ is just, for three of the four components of A_μ,

$$\dot{A}_i - \frac{\delta H}{\delta p^i} = \dot{A}_i - (p^i + A_{0,i}) = \dot{A}_i + A^0{}_{,i} - p^i = -E_i - p^i = 0. \tag{13}$$

What one reckons as gauge freedom must be compatible with this on-shell relationship. While p^i has vanishing Poisson brackets with each first-class constraint separately, E_i is invariant under a transformation of A_μ only if one tunes the primary and secondary constraints' smearing functions to cancel out the induced changes in E_i. Thus being a gauge transformation requires more than leaving p^i alone (as one might think sufficient if one gives the Hamiltonian formalism priority [16] [7, p. 20]); it requires leaving E_i alone as well. Otherwise one makes the relationship $\dot{A}_i - \frac{\delta H}{\delta p^i} = -E_i - p^i = 0$ gauge-dependent, spoiling Hamiltonian-Lagrangian equivalence and undermining the physical meaning of p^i on-shell (the only context where p^i has any physical meaning). These concerns about the extended Hamiltonian bear some resemblance to Sugano, Kagraoka and Kimura's [38].

7 Counting Degrees of Freedom

One might think that correct counting of degrees of freedom would depend on whether one takes the generator of gauge transformations to be a special combination of the first-class constraints or an arbitrary combination. In the former case, there are only as many independent functions of time (and perhaps space) as there are *primary* first-class constraints; some of the constraints are smeared with the time derivative of functions that smear other constraints. In the latter case there are as many independent functions of time (and perhaps space) as there are first-class constraints. However, behavior over time is irrelevant; hence a function and its time derivative, being independent *at a moment*, count separately. Thus the counting works out the same either way [7, pp. 89, 90]. Getting the correct number of degrees of freedom thus does not show whether each first-class constraint or only the special combination G generates gauge transformations.

8 Error in Identifying Primaries As Generating Gauge Transformations

One major reason that first-class constraints wrongly have been thought to generate gauge transformations is that Dirac claims to prove it early in his book [5, p. 21]. One finds the same proof repeated in other works [7, 10, 34]. The canonical Hamiltonian is, up to a boundary term [35, p. 127],

$$\int d^3x[\frac{1}{2}(p^i)^2 + \frac{1}{4}F_{ij}^2 - A_0 p^i{}_{,i}]. \tag{14}$$

The total Hamiltonian adds the primary constraint with an arbitrary velocity. Dirac, not using the gauge generator G, saw the arbitrary velocities v multiplying the primaries outside his H' but apparently forgot the corresponding arbitrary q's (like A_0) multiplying the secondaries *inside* H'. Thus he did not notice that the first-class primaries outside H' and first-class secondaries inside H' work as a team to generate gauge transformations. Thus

> [w]e come to the conclusion that the ϕ_a's, which appeared in the theory in the first place as the primary first-class constraints, have this meaning: *as generating functions of infinitesimal contact transformations, they lead to changes in the q's and the p's that do not affect the physical state.* [5, p. 21, emphasis in the original]

One could hardly reach such a conclusion without thinking that the primaries were the locus of all dependence on the arbitrary functions. He then conjectures that the same holds for first-class secondary constraints. As appeared above, neither the primaries nor the secondaries generate a gauge transformation in electromagnetism. Dirac's failure presumably encouraged him to extend the Hamiltonian in order to recover what was apparently missing [5, pp. 25, 31]. But it is unnecessary and obscures the relation of the fields to those in the more perspicuous and reliable Lagrangian formalism [41, p. 39]. Indeed the extended Hamiltonian breaks Hamiltonian-Lagrangian equivalence [42]. Requiring Hamiltonian-Lagrangian equivalence fixes the supposed ambiguity permitting the extended Hamiltonian [43].

8.1 Perpetuation in Recent Works

This same mistake continues to be made, as in ([7, 10, 34]). The problem will be clearer if one starts with Wipf's treatment. The time evolution of a system with first-class constraints is derived from the total/primary Hamiltonian H_p (the canonical Hamiltonian H plus the primary constraints ϕ_a with arbitrary multiplier functions μ^a). For a phase space quantity F, one compares

> two infinitesimal time evolutions of $F = F(0)$ given by H_p with different values of the multipliers,
>
> $$F_i(t) = t\{F, H\} + t\{F, \phi_a\}\mu_i^a \qquad i = 1, 2 \qquad . \tag{5.18}$$
>
> The difference $\delta F = F_2(t) - F_1(t)$ between the values is then
>
> $$\delta_\mu F = \{F, \mu^a \phi_a\}, \qquad , \qquad \mu = t(\mu^2 - \mu^1). \tag{5.19}$$
>
> Such a transformation does not alter the physical state at time t, and hence is called *gauge transformation* [reference to Dirac's book [5]] [34, p. 27]

Like Dirac, Wipf has overlooked the fact that the canonical Hamiltonian also is influenced by the multiplier functions: the canonical Hamiltonian contains A_0 multiplying the secondary constraint, while the multiplier function is $\dot{A}_0$. Thus not only the μ^a multiplier functions, *but also the canonical Hamiltonian H*, needs a subscript 1 or 2. With this mistake corrected, one has

$$\begin{gathered} \delta_\mu F = t\{F, H_2 - H_1\} + t\{F, \phi_a\}(\mu_2^a - \mu_1^a) = \\ t\{F, \int d^3y - (A_0^2 - A_0^1)(y)\pi^i{}_{,i}(y)\} + t\{F, \int d^3y p^0(y)\}(\mu_2 - \mu_1). \end{gathered} \tag{15}$$

The correct expression exhibits the secondary constraint(s) working together with the primary constraint(s). Given the Dirac-Wipf erroneous expression involving only the primary constraint, a 'gauge transformation' that changes only A_0 would be exhibited. But as was shown in detail above, or as follows from a moment of reflection on electrostatics, changing A_0 while leaving everything else alone *does* alter the physical state, and hence is not a gauge transformation. It is obvious that this expression does not change the canonical momenta p^0 or p^i; what does it do to A_ν? The corrected expression, unlike Dirac's, changes A_j as well, as it should. Letting $F = A_\nu(x)$ gives (changing notation from t to δt for a small interval, and recalling that our initial moment can be called $t = 0$)

$$\begin{gathered} \delta_\mu A_\nu(\delta t, x) = \\ \delta t\{A_\nu(0, x), \int d^3y - (A_0^2 - A_0^1)(0, y)\pi^i{}_{,i}\} + \delta t\{A_\nu, \int d^3y p^0\}(\mu_2 - \mu_1) = \end{gathered}$$

$$\delta t \int d^3y \delta^i_\nu \delta(x,y)(A^2_{0},_i - A^1_{0},_i)(y) + \delta t \delta^0_\nu (\mu_2 - \mu_1)(x) =$$
$$\delta t \delta^i_\nu (A^2_{0},_i - A^1_{0},_i)(x) + \delta t \delta^0_\nu (\dot{A}^2_0 - \dot{A}^1_0)(0,x) =$$
$$\delta t (A^2_0 - A^1_0),_\nu (0,x). \qquad (16)$$

This expression clearly resembles the usual gauge transformation property of electromagnetism $-\partial_\nu \epsilon$, so one can say that the two evolutions differ by a (standard) gauge transformation, as one would hope. Thus it is false that the primary first-class constraints generate a gauge transformation in examples like electromagnetism, because it is a special combination of the primaries and secondaries that does so. The primary by itself changes $\vec{E}$, as does the secondary by itself. Continuing with Wipf,

> [w]e conclude that the most general physically possible motion should allow for an arbitrary gauge transformation to be performed during the time evolution. But H_p contains only the primary FCC. We thus have to add to H_p the secondary FCC multiplied by arbitrary functions. This led Dirac to introduce the *extended Hamiltonian*... which contains *all* FCC [reference to Dirac's book [5]]. [34, p. 28]

But the secondary first-class constraint already is present in the total Hamiltonian, as is the gauge freedom, so there is nothing missing that needs adding in by hand. Such an omission is all the more consequential in relation to General Relativity, in which the canonical Hamiltonian is *nothing but* secondary constraints (and boundary terms).

Now the problem in the treatment of Henneaux and Teitelboim can be identified readily and treated briefly.

> Now, the coefficients v^a are arbitrary functions of time, which means that the value of the canonical variables at t_2 will depend on the choice of the v^a in the interval $t_1 \leq t \leq t_2$. Consider, in particular, $t_1 + \delta t$. The difference between the values of a dynamical variable F at time t_2, corresponding to two different choices v^a, $\tilde{v}^a$ of the arbitrary functions at time t_1, takes the form
>
> $$\delta F = \delta v^a [F, \phi_a] \qquad (1.35)$$
>
> with $\delta v^a = (v^a - \tilde{v}^a)\delta t$. Therefore the transformation (1.35) does not alter the physical state at time t_2. We then say, extending a terminology used in the theory of gauge fields, that *the first-class primary constraints generate gauge transformations.* [7, p. 17]

By now it has been seen that this statement is false. But its proximate cause is evident after looking at Wipf's treatment, namely, neglecting the fact that the secondary constraint was already present in the canonical Hamiltonian with a gauge-dependent coefficient in the form $\int d^3y - A_0(y)p^i{}_{,i}(y)$. They give just Dirac's argument again. But it simply isn't the case that the difference between the values of the two evolutions is given by $\delta F = \delta v^a[F, \phi_a]$, because the two evolutions are also influenced by their differing terms of the form $\int d^3y - A_0(y)p^i{}_{,i}(y)$ or the like from the canonical Hamiltonian, at least for theories with secondary first-class constraints like electromagnetism, Yang-Mills, and GR.

Unfortunately Dirac's mistake also reappears in the recent book by Rothe and Rothe [10, p. 68]. Failure to look inside the black box H, the canonical Hamiltonian, and see the secondary first-class constraints while doing this little calculation seems to be the cause. Choosing A_i as a phase space quantity to test the behavior of the quantity built from primary first-class constraints gives an easy diagnostic to see that no gauge transformation is generated.

9 Dirac Conjecture's Presupposition

Dirac, having supposedly shown that primary first-class constraints generate gauge transformations, conjectured that secondary first-class constraints do the same [5]. Eventually it was found that this conjecture has counterexamples, namely ineffective constraints, though they are a bit exotic and might sensibly be banned [7]. But the Dirac conjecture has a much more serious problem, namely, the falsehood of its presupposition that primary first-class constraints generate gauge transformations. Whether that problem makes the Dirac conjecture false or lacking in truth value will depend on the logical details of the formulation, but it certainly winds up not being an interesting truth. Complementing the falsification by direct calculation above is a diagnosis (just above) of the mistake that Dirac and others have made in failing to pay attention to the term $\int d^3x - A_0 p^i{}_{,i}$ term in the Hamiltonian.

How does one reconcile this result that a primary first-class constraint does not generate a gauge transformation with the multiple 'proofs' of the Dirac conjecture in the literature [7, 44–46] and the statements that it can be made true by interpretive choice [6, 7]? These proofs usually presuppose that a Dirac-style argument has already successfully addressed primary first-class constraints, so the only remaining task involves secondary or higher order constraints. The remaining task tends to involve statements about first-class constraints, which are simply *assumed* to generate gauge transformations individually. Thus 'proofs' of the Dirac conjecture are frequently just statements about Poisson brackets and first-class secondary (and higher) constraints—straightforward technical questions with results that are, presumably, correct. Conceptually involved proofs of the Dirac conjecture, which essentially talk about gauge transformations, must fail. But mere technical statements about vanishing Poisson brackets are not threatened at all. Hence there is no tension with the correctness of the calculations.

10 Observability of P^i *vs.* E_i Can Be Crucial

While it is acknowledged that the extended Hamiltonian not equivalent to L strictly, this inequivalence is often held to be harmless because they are equivalent for "observables." This claim presumably is intended to mean that the extended Hamiltonian is *empirically equivalent* to L, differing only about unobservable matters. Such a response will be satisfactory only if "observable" here is used in the ordinary sense of running experiments. Technical stipulations about the word "observable," especially distinctively Hamiltonian stipulations, are irrelevant. Unfortunately it is not the case that the extended Hamiltonian is empirically equivalent to the Lagrangian, a fact that has been masked by equivocating on the word "observable" between the ordinary experimental sense and a technical Hamiltonian sense. It is peculiar to think of observing canonical momenta conjugate to standard Lagrangian coordinates—in fact it seems to be impossible to observe that kind of canonical momentum as such. What would be the operational procedure for observing p^i? Rather, its experimental significance is purely on-shell, parasitic upon the observability of suitable functions of q^i and/or derivatives of q^i—derivatives (spatial and temporal) of A_μ in the electromagnetic case. One neither acquires new experimental powers (such as the ability to sense canonical momenta) nor loses old ones (such as the ability to detect a certain combination of derivatives of A_μ) by changing formalisms from the Lagrangian to the Hamiltonian. There are two ways to see that p^i is not the primordial observable electric field. The first way involves the fact that p^i does not even appear as an independent field in the Lagrangian formalism, which formalism is correct and transparent. While it is perfectly acceptable for some quantity to be introduced that is on-shell equivalent to the Lagrangian electric field, there is no way for that new quantity to become the electric field primordially, rather than merely derivatively and on-shell. A_μ or a function of its derivatives still has that job. Apart from constraints, canonical momenta are auxiliary fields in the Hamiltonian action $\int dt(p\dot{q} - H(q,p))$: one can vary with respect to p, get an equation $\dot{q} - \frac{\delta H}{\delta p} = 0$ to solve for p, and then eliminate p to get $\int dtL$. One would scarcely call an auxiliary field a primordial observable and the remaining q in L derived! The second way involves the fact that the electric field is what pushes on charge; but it is easy to see that in both the Lagrangian and Hamiltonian contexts, what couples to the current density is not p^i, but A_μ. For a complex scalar field ψ, the Lagrangian interaction term takes the form $\sim (\psi \partial_\alpha \psi^* - \psi^* \partial_\alpha \psi) A^\alpha + \psi \psi^* A^2$. The absence of terms connecting ψ with derivatives of A_μ implies that charge couples to A_μ and/or its derivatives, not to the canonical momenta conguate to A_μ, even in the Hamiltonian context. What is the operational procedure for measuring p^i? The only plausible answer is to use on-shell equivalence to the empirically available F_{0i}, which involves derivatives of A_μ. Otherwise, what reason is there to believe that any procedure for measuring p^i involves a measurement of the quantity that pushes on charge? Thus one *should* be disturbed, *pace* Costa *et al.* [16], by the failure of $\dot{A}_i = \frac{\delta H_E}{\delta p_i}$ to return the usual Lagrangian relation between p^i and the derivatives of A_μ from the extended Hamiltonian. The coupling of charge-current to A_μ ensures that A_μ or something built from its derivatives is the primordial observable electric field. Thus the usual argument [7, 10, 16] to show that the inequivalence of the extended Hamiltonian to the Lagrangian is harmless because irrelevant to observable quantities, fails; unless "observables" are taken in the ordinary empirical sense, rather than a technical Hamiltonian sense, empirical equivalence is not shown.

The 'proof' of the Dirac conjecture by Costa *et al.* [16] deserves special comment. This paper goes beyond other treatments of the supposed equivalence of the extended Hamiltonian to the total Hamiltonian for observables [7, 10] in explicitly addressing the example of electromagnetism in sufficient detail. The equivalence conclusion is reached by explicitly taking the canonical momentum p^i to be the primordial physically meaningful quantity playing the role of the electric field. For a function of canonical coordinates and momenta (no time derivatives), having vanishing Poisson bracket with the gauge generator requires having vanishing Poisson bracket with each first-class constraint, because different orders of time derivative of the smearing function cannot cancel each other out [16]. But that latter condition opens the door to taking all first-class constraints to generate gauge transformations and using the extended Hamiltonian, they claim. They recognize that one can use Hamiltonian's equations from the total Hamiltonian and find a quantity that is equal in value on-shell to a gauge-invariant function of q and p. I observe that the electric field is in this category. They also observe that such a quantity is invariant under the gauge generator of the total Hamiltonian (the specially tuned combination of first-class constraints) and is not invariant under the first-class constraints separately, as I emphasized above. In their words,

> [o]ne can verify the invariance under [the usual electromagnetic gauge transformation of A_μ] of the equations of motion ...
>
> $$\partial^0 A^j = \pi_j + \partial^j A^0, \qquad (3.8b)$$
>
> ... deriving from the total Hamiltonian....
>
> We next recognize F^{ij}, π_j ... [matter terms suppressed] as the canonical forms of the basic gauge-invariant quantities of electrodynamics. One can easily check that all these functions are indeed first class. Thus, F^{ij}, π_j... are also invariant under the extended infinitesimal transformations [generated by an arbitrary sum of independently smeared first-class constraints]. ... [That extended first-class transformation] leaves invariant the equations of motion...
>
> $$\partial^0 A^j = \pi_j + \partial^j A^0 - \partial^j \xi^2, \qquad (3.12b)$$
>
> ... arising from the extended Hamiltonian
>
> $$H_E = H + \int d^3x \{\xi^1(\mathbf{x})\pi_0(\mathbf{x}) + \xi^2(\mathbf{x})[\partial^j \pi_j(\mathbf{x}) - \ldots]\}. \qquad (3.13)$$
>
> [spinor contribution in secondary constraint suppressed]
> Here ξ^1 and ξ^2 are arbitrary Lagrange multipliers.
>
> As a matter of fact, the sets of equations of motion (3.8) and (3.12) are *different*. However, irrespective of whether one starts from (3.8) or (3.12) one arrives at the Maxwell equations
>
> $$\partial^0 F^{ij} = \partial^i \pi_j - \partial^j \pi_i, \qquad (3.14)$$
> $$\partial^0 \pi_j = \partial^i F^{ij} \ldots, \qquad (3.15)$$
>
> [16, pp. 407, 408]

I note the absence of Gauss's law!

They continue:

> Therefore, H_T and H_E generate the same time evolution for the gauge-invariant quantities, as required by [the equation of motion for gauge invariant phase space functions].

> We now discuss the alternative formalism-dependent realizations of the electric field $(-\pi_j)$. From (3.8b) one obtains
>
> $$\pi_j = F^{0j}. \tag{3.17}$$
>
> Hence, F^{0j} is a faithful realization of π_j within the formalism of the total Hamiltonian. We can check that F^{0j} is invariant under [the gauge generator related to the total Hamiltonian, which combines the first-class constraints with related smearings] but not under [the sum of separately smeared first-class constraints, which is related to the extended Hamiltonian formalism]. [16, p. 408]

This is the crucial point announced in my paper's title—but Costa *et al.* fail to recognize the absurdity of the results of the extended Hamiltonian formalism. They continue:

> On the other hand, the formalism of the extended Hamiltonian provides the equally faithful realization for π_j [see Eq. (3.12b)]
>
> $$\pi_j = F^{0j} + \partial^j \xi^2, \tag{3.18}$$
>
> which is invariant under [the sum of independently smeared first-class constraints]. One should not be puzzled by the fact that (3.18) does not coincide with (3.17) or, what amounts to the same thing, with the Lagrangian definition of π_j .... [16, p. 408]

But one *should* be puzzled. If π_j is equated to the electric field (as they say), and if F^{0j} is just an abbreviation for a familiar expression involving derivatives of A_μ (as follows from (3.12b) and (3.18)—and hence is still the electric field!), then we have the contradiction (electric field = electric field + arbitrary gradient). With this contradiction in hand, one can derive various other plausible errors. This arbitrary gradient is what spoiled Gauss's law above. In any case F^{0j} has a much better claim to be the electric field than does π_j, which is just an auxiliary field in the Hamiltonian action. Thinking that functions of phase space were the only quantities that needed to stay gauge invariant—that is, not considering the actual electric field—is what opened the door to the extended Hamiltonian and taking each first-class constraint as separately generating a gauge transformation. One should infer that an isolated first-class constraint does not generate a gauge transformation in electromagnetism. F^{0j} is the primordial observable electric field; the canonical momentum as an independent field is formalism-dependent, not even appearing in the Lagrangian formalism. In a

Lagrangian for charged matter with an electromagnetic field, charge-current couples primordially to A_μ, from which $\vec{E}$ is derived, and not to the canonical momentum. Velocities (such as appear in the electric field) are not physically recondite—automobiles have gauges that measure them—but canonical momenta are: they acquire physical significance solely on-shell, as Costa *et al.* remind us. Hence failure to recognize the fundamentality of the Lagrangian formalism leads them to claim to have vindicated the Dirac conjecture, when they had all the ingredients and calculations necessary to refute it instead.

One might also worry that physically meaningful quantities are expected to have *vanishing* Poisson bracket with the gauge generator [16], given that tensors in GR will not qualify due to the Lie derivative term. (This problem is peculiar to external symmetries.) While this requirement is not unusual, it introduces the difficulties afflicting the notion of observables in GR into the presumably more perspicuous discussion of equivalence of equations of motion.

Crucial to gauge-transforming the electric field (as opposed to the canonical momentum to which it is equal on-shell) is having a gauge transformation formula for velocities. In a Hamiltonian formalism it is tempting, though inadvisable, to avoid velocities in favor of functions of q and p. But the Lagrangian formalism essentially involves the commutativity of gauge variation and time differentiation [47, 48]. Imposing that condition in the Hamiltonian formalism using the total Hamiltonian (the one equivalent to the Lagrangian) yields the gauge generator G [47, 48]. Thus the Hamiltonian formalism naturally can give the correct gauge transformation for velocities and quantities built from them, such as the electric field. One does not need to avoid looking for gauge-invariant quantities involving the velocities and default to functions of only q and p in a Hamiltonian context, as Costa *et al.* did [16]. Alternately, one can be satisfied in a (total) Hamiltonian formalism with functions of q and p [49] but, in view of the need to preserve Hamiltonian-Lagrangian equivalence, avoid seeking the largest collection of transformations (the first-class transformations rather than just the gauge generator G) that preserve the phase space quantities at the expense of Hamiltonian-Lagrangian equivalence.

11 Anderson and Bergmann (1951): Canonical Transformations and Lagrangian-Equivalence

None of this confusion associated with Hamiltonian transformations that aren't induced by Lagrangian gauge transformations should be much of a surprise, ideally, in that Anderson and Bergmann explicitly discussed how the preservation of the Lagrangian constraint surface, which they called Σ_l, corresponds to canonical transformations generated by the gauge generator G [1]. Hence one would expect transformations that aren't generated by G—*e.g.*, those generated by an isolated primary constraint in a theory (such as Maxwell's electromagnetism or GR) where the gauge generator G doesn't contain that primary constraint in isolation (*i.e.*, smeared by its very own arbitrary function)—not to preserve the Lagrangian constraint surface. Hence the point that a first class constraint by itself (in theories where such does not appear in isolation in G) generates not a gauge transformation, but a violation of the usual Lagrangian constraint surface, is already implicit in Anderson and Bergmann—at least if one is working with canonical transformations. (Outside the realm of canonical transformations, one can still take Poisson brackets directly. But then there are far fewer rules and hence there is much less reason to expect anything good to happen.) As they observe,

> Naturally, other forms of the hamiltonian [*sic*] density can be obtained by canonical transformations; but the arguments appearing in such new expressions will no longer have the significance of the original field variables y_A and the momentum densities defined by Eq. (4.2) [which defines the canonical momenta as $\pi^A \equiv \frac{\partial L}{\partial \dot{y}_A}$]. It follows in particular that transformations of the form (2.4) ["invariant" transformations changing $\mathcal{L}$ by at most a divergence, such as electromagnetic gauge transformations or passive coordinate transformations in GR] will change the expression (4.9) [for the Hamiltonian density] at most by adding to it further linear combinations of the primary constrains, i.e., by leading to new arbitrary functions w^i. [1, p. 1021]

So they invented the gauge generator G to make sure that the q's and p's keep their usual meanings.

Unfortunately the point was lost after Bergmann, Anderson and Dirac repeatedly said things that were incompatible with that correct claim about the gauge generator G, namely, that a first-class constraint generates a gauge transformation. Accounting for the change in Bergmann's and Anderson's view is beyond the scope of this paper. It seems to be, at least in part, connected with the tendency to drop the primary constraints and their associated canonical coordinates from the phase space, especially once the primary constraints for GR were expressed in the trivial form of the vanishing of some momenta. The view that a first-class constraint generates a gauge transformation then became the conventional wisdom expressed in countless works for decades, with lingering consequences (such as regarding observables [50, 51]) even where the gauge generator has been gaining ground.

11.1 Canonical Transformations Generating Position-Dependent Field Redefinitions

If one wishes, one can treat a smeared primary constraint as a canonical transformation generator in the sense of ([1, 52]) and preserve *some* sense of physical equivalence for the transformation generated by the primary first-class constraints. That is a feature of dynamics in general, not Dirac-Bergmann constrained dynamics in particular. It makes use of p^0, but not the fact that $p^0 = 0$ (its being a constraint) or its having vanishing Poisson brackets with the other constraints and Hamiltonian (its being first-class). But equivalence is preserved only by losing some of the original fields' meanings.

Let $C = \int d^3y \epsilon(t, y) p^0(y)$. One can add to the Hamiltonian action the time integral of the total time derivative of this quantity. One gets new canonical coordinates, $Q^A = q^A + \frac{\delta C}{\delta p_A}$, and new canonical momenta, $P_A = p_A - \frac{\delta C}{\delta q^A}$, and a slightly altered Hamiltonian, $K = H + \frac{\partial C}{\partial t} = H + \int d^3y p_0 \frac{\partial \epsilon}{\partial t}$, which adds a term proportional to a primary constraint only. Of the new Q's, only the 0th differs from the old q's ($Q^0 = q^0 + \epsilon$); the new momenta are the same as the old. The trouble arises subtly: for the other Q's velocity-momentum relation, $\dot{Q}^a = \frac{\delta K}{\delta P_a}$, the dependence on the 0th canonical coordinate in K involves the altered Q^0. The electromagnetic scalar potential is involved in the relation between $\dot{A}_i$ and p^i, so changing the scalar potential alters the relationship between the canonical momenta and the velocities, the sort of issue to which Anderson and Bergmann called attention. For q^0 corresponding to A_0 (or the lapse N or shift vector β^i in General Relativity), one can change q^0 *alone* however one likes over time and place (which is what the corresponding primary constraint does)—but only at the cost of ceasing to interpret the new canonical coordinate $Q^0 = q^0 + \delta q^0$ as (minus[4]) the scalar potential A_0 (or lapse N or shift β^i)! The new Hamiltonian K

[4] I use $- + ++$ metric signature. But indices are placed up and down freely, depending on whether the general paradigm Q^A or the specific case A_μ is more relevant.

differs from H only by a term involving a primary constraint $p_0 = P_0$, which doesn't matter. The new velocity-momentum relationship is

$$\dot{Q}^i = \frac{\delta K}{\delta P_i} = \frac{\partial}{\partial P_i}(\frac{1}{2}P_j^2 + \frac{1}{4}F_{jk}^2 + P_j \partial_j [Q^0 - \epsilon]) = P_i + \partial_i (Q^0 - \epsilon). \tag{17}$$

One can solve for P_i and then take the 3-divergence:

$$P_{i,i} = \partial_i (\dot{Q}^i - Q^0{}_{,i} + \epsilon_{,i}) = \partial_i (\dot{q}^i - \partial_i q^0) = \partial_i F_{0i} = -\partial_i E_i. \tag{18}$$

By using the full apparatus of a canonical transformation and keeping track of the fact that Q^0 is no longer (up to a sign) the electromagnetic scalar potential as q^0 is, one can resolve the contradiction about vanishing *vs.* nonvanishing divergence of the canonical momentum *vis-a-vis* the electric field. Such reinterpretation, which strips the new canonical coordinates of some of their usual physical meaning and replaces them with a pointlessly indirect substitute, though mathematically permitted, *is certainly not what people usually intend when they say that a first-class constraint generates a gauge transformation.* What they mean, at least tacitly, is that the fields after the transformation by direct application of Poisson brackets (not a canonical transformation) have their usual meaning—hence one would (try to) calculate the electric field from $\dot{Q}^i - Q^0{}_{,i}$ (thus spoiling the Lagrangian constraints, as shown above) rather than $\dot{Q}^i - Q^0{}_{,i} + \epsilon_{,i}$. Supposing that one attempts to retain the old connection between the 0th canonical coordinate and the electromagnetic scalar potential, one can calculate the alteration in the electric field (that is, the electric field from Q^A less the electric field from q^A) as $\delta F_{0n} = \partial_0 \delta A_n - \partial_n \delta A_0 = 0 - \partial_n \frac{\delta C}{\delta p_0} = -\partial_n \epsilon$, as found above by more mundane means. To avoid the contradiction of a physics-preserving transformation that changes the physics, one can and must re-work the connection between Q^0 and A_0, as shown. But simply avoiding this sort of generating function, one that is not (a special case of) G, is more advisable.

In short, as a canonical transformation generator with suitable smearing, p_0, the primary first-class constraint, generates only an *obfuscating position-dependent change of variables.* It has nothing to do with the usual gauge freedoms of electromagnetism (or GR, by analogy). It has nothing to do with p_0's being first-class; the canonical transformation would work equally well for Proca's massive electromagnetism, in which that constraint is second-class. Only in detail does it even depend on p_0's being a constraint, as opposed to merely something that lives on phase space. It is easy to see reasons not to make such transformations, and wrong to make them without understanding what they do.

One can also try the secondary constraint $p^i{}_{,i}$ as a generator of a canonical transformation: $D = \int d^3y - \epsilon_{,i}\, p^i(y)$ after dropping a boundary term. The new canonical coordinates are $Q^A = q^A + \frac{\delta D}{\delta p_A} = q^A - \epsilon_{,i}\, \delta^i_A = A_\alpha - \epsilon_{,i}\, \delta^i_\alpha$. The new canonical momenta are $P_A = p_A - \frac{\delta D}{\delta q^A} = p_A$. One sees that the new Q^i are not the original electromagnetic 3-vector potential A_i anymore. (They are not a gauge-transformed vector potential, either, unless one throws the trouble onto Q^0 by stripping it of its relation to the electromagnetic scalar potential.) The new Hamiltonian is $K = H + \frac{\partial D}{\partial t} = H + \int d^3y - p^i \epsilon_{,0i}$, which differs from the old by a term proportional to the secondary constraints (and perhaps a boundary term). Thus the altered $\dot{Q} - P$ relation is $\dot{Q}^i = \frac{\delta K}{\delta P_i} = P_i + Q^0{}_{,i} - \epsilon_{,0i}$. One can take the divergence and solve for $P^i{}_{,i}$: $P^i{}_{,i} = \partial_i(Q^i{}_{,0} - Q^0{}_{,i} + \epsilon_{,0i}) = \partial_i(q^i{}_{,0} - \partial_i q^0) = \partial_i F_{0i} = -\partial_i E_i$. By taking into account the fact that the new Q's are no longer all just the electromagnetic 4-vector potential A_μ, one resolves the contradiction between vanishing and nonvanishing divergence. The electric field $\vec{E}$, which is an observable by any reasonable standard, is no longer specified simply by (derivatives) of the new canonical coordinates Q, but requires the arbitrary smearing function ϵ used in making the change of field variables also. That is permissible but hardly illuminating.

One can do basically the same thing with Proca's massive electromagnetism [17, 35, 44], taking the secondary constraint, now second-class, as the generator of a canonical transformation. The secondary sprouts a new piece $m^2 A_0$. The transformed massive Hamiltonian K gets an extra new term $m^2 Q^0 \dot{\epsilon}$. The new canonical momenta reflect a change in the primary constraint form: $P_0 = p_0 - m^2 \epsilon$. But everything cancels out eventually, leaving equations equivalent to the usual ones for massive electromagnetism, naturally. Only in detail does the first-class (massless) *vs.* second-class (massive) character of the secondary constraint make any difference. As the generator of a canonical transformation, a first-class constraint doesn't generate a gauge transformation in massless electromagnetism any more than a second-class constraint generates a gauge transformation in massive electromagnetism. Both generate permissible but pointless field redefinitions.

The key difference is that a special combination of first-class constraints in massless electromagnetism does generate a gauge transformation, whereas in massive electromagnetism, there is no gauge transformation to generate, so no combination of anything can generate one. Amusingly, given that the key issue is changing A_μ by a four-dimensional gradient, and not directly the first-class or even constraint character of the generator, one can use the same special sum $\int d^3y[-p^0(y)\dot{\epsilon}(t,y) + p^i{}_{,i}(y)\epsilon(t,y)]$ as applied to massive electromagnetism to generate a Stueckelberg-like gauged version of massive electromagnetism, with the smearing function ϵ, in this case not varied in the action, as the gauge compensation field. $\int d^3y[-p^0(y)\dot{\epsilon}(t,y) + p^i{}_{,i}(y)\epsilon(t,y)]$ is no longer a sum of constraints (not even second-class ones, though p^0 is a second-class constraint). This possibility might take on some importance in application to installing artificial gauge freedom in massive Yang-Mills theories, where the proper form has been a matter of some controversy [53–58].

Finally, one can use the gauge generator G as the generator of a canonical transformation in Maxwell's electromagnetism. It turns out that, in contrast to an arbitrary function on phase space (or a first-class constraint) as a generator, the gauge generator G generates the *very same thing* for the canonical variables as a canonical transformation as it does 'by hand' by taking the Poisson bracket directly with q and p. Dropping a spatial divergence, one has $G = \int d^3x - \epsilon_{,\mu} p^\mu$. One gets the new canonical coordinates $Q^A = q^A + \frac{\delta G}{\delta p_A} = A_\alpha - \epsilon_{,\alpha}$ and new canonical momenta $P_A = p_A - \frac{\delta G}{\delta q^A} = p_A$, and a slightly altered Hamiltonian, $K = H + \frac{\partial G}{\partial t} = H + \int d^3y - p^\mu \epsilon_{,\mu 0}$, which adds related terms proportional to the primary and secondary constraints (and a spatial boundary term). Significantly, $Q^A - q^A = \frac{\delta G}{\delta p_A} = \{q^A, G\}$ and $P_A - p_A = -\frac{\delta G}{\delta q^A} = \{p_A, G\}$. That is, G does the very same thing to q^A and p_A whether one simply takes the Poisson bracket with G directly or uses G to generate a canonical transformation. Thus if one uses G, one can be nonchalant (as people often are using first-class constraints separately [5, p. 21]) about whether one is making a canonical transformation or is merely directly taking a Poisson bracket; that lack of concern does not carry over to expressions different from G, however. G does one good thing, recovering the usual electromagnetic gauge transformations, used either way. By contrast, each isolated first-class constraint offers a choice of two bad things (one disastrous, one merely awkward): it can either destroy the field equations if used directly in Poisson brackets, or generate a confusing change of physical meaning of the variables as the generator of a canonical transformation.

One can summarize in a table some of the results about using the gauge generator G *vs.* a smeared individual constraint or other phase space function, and using it as a canonical transformation generating function *vs.* using it directly *via* Poisson bracket. Presumably the experience for electromagnetism largely carries over for other constrained theories. For the first-class theory one has these phenomena:

	Canonical transformation	Direct Poisson bracket
Gauge generator G	Gauge transformation	Gauge transformation
Smeared constraint	Locally varying field redefinition	Spoils $\vec{\nabla} \cdot \vec{E} = 0$

The entries in the first column can be described in more detail. One can illustrate the illuminating (invariant) canonical transformations related to G (top left corner) and the obscuring but permissible more general canonical transformations (bottom left corner) in the following diagrams.

The first is a commutative diagram with well understood entries and transformations. (The equation numbers correspond to the remarks in Anderson and Bergmann [1].)

$$
\begin{array}{ccc}
\mathcal{L} & \xrightarrow[\delta\mathcal{L}=div,\ \delta A_\mu=\partial_\mu\xi,\ \delta g_{\mu\nu}=\pounds_\xi g_{\mu\nu}]{\textit{invariant gauge } 2.4:} & \mathcal{L}' \\
\textit{constrained Legendre}\Big\downarrow & & \Big\downarrow\textit{constrained Legendre} \\
\mathcal{H} & \xrightarrow[\textit{preserves } q_A \textit{ sense},\ 4.2:\ \pi^A=\frac{\partial\mathcal{L}}{\partial\dot{q}_A}]{\textit{invariant canonical } G} & \mathcal{H}'
\end{array}
$$

One can of course also make point transformations, changes among the q_A's only. In electromagnetism, one might use A^μ instead of A_μ; that is probably the least bad choice if one does not stick with A_μ. In GR one is free to use $g_{\mu\nu}$, $\mathfrak{g}^{\mu\nu}$ (which equals $g^{\mu\nu}\sqrt{-g}$), or various other fields, for example. For Anderson and Bergmann, this freedom to make point transformations is already implied by their rather abstract use of q_A (or actually y_A in their notation) and rather general form of gauge transformations. A field redefinition from one choice of q_A to another will of course induce a contragredient change in the canonical momenta. One can also add a divergence to the Lagrangian density. Such an alteration will also tend to alter the canonical momenta, but not mysteriously. These two changes were combined to simplify the primary constraints of GR in 1958 [23, 24]. One could augment the diagram above to indicate more fully the resources of Lagrangian field theory. The main point, however, is to distinguish adequately what is allowed within the Lagrangian formalism from the greater, and more dangerous, generality of the Hamiltonian formalism.

The second is an unhealthy aspiring commutative diagram illustrating how allowing general canonical transformations—for example, a single primary or secondary first-class constraint—leads to entries and transformations that are not widely understood, if meaningful at all.

$$
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{?} & \mathcal{L}' \\
\textit{constrained Legendre}\Big\downarrow & & \Big\uparrow\textit{inverse constrained Legendre?} \\
\mathcal{H} & \xrightarrow[\textit{violates } q_A \textit{ sense or } 4.2:\ \pi^A=\frac{\partial\mathcal{L}}{\partial\dot{q}_A}]{\textit{general canonical}} & \mathcal{H}'
\end{array}
$$

A canonical transformation to action-angle variables, for example, would give a Hamiltonian that might not readily admit an inverse Legendre transformation back to a Lagrangian. Suffice it to say that Hamiltonian-Lagrangian equivalence is obscured by general canonical transformations. It is not very obvious what the resulting equations mean physically, given that the usual Lagrangian variables such as $g_{\mu\nu}$, not the canonical momenta, are the ones with known direct empirical meaning. General canonical transformations are useful tricks in mechanics, where one already understands what everything means, but needs to solve specific problems. But a position-dependent change of variables when one is already on marshy ground, having difficulty identifying change or observables, is inadvisable without the greatest care.

12 How to Get Right Electromagnetic Fields with Wrong Gauge Transformations

One might think that misidentifying the generator of a gauge transformation would lead to selecting the wrong fields in mildly nontrivial examples such as electromagnetism. That a first-class constraint generates a gauge transformation was held by Bergmann and collaborators [2–4], not just Dirac [5]. Bergmann commented that, for electromagnetism, the physical variables are (omitting sources, unlike him) $\nabla \times \vec{E}$ and $\nabla \times \vec{B}$ because they are neither 0 nor gauge-dependent [2]. Bergmann evidently got the right fields for electromagnetism. How is that result compatible with his having the wrong generator(s)?

Using his condition of vanishing Poisson bracket with each first-class constraint, one should find that $\nabla \cdot \vec{E}$ is gauge-dependent but $\nabla \times \vec{E}$ is gauge-invariant; $\vec{B}$ is gauge-invariant, but $\nabla \cdot \vec{B} = 0$. That $\nabla \cdot \vec{E}$ is gauge-dependent is incredible, but it is tempting not to do the calculation because the expected answer is obvious. By contrast, using G [1], one finds that $\vec{E}$ is gauge-invariant, as is $\vec{B}$, but both have vanishing divergence. One keeps the same fields, but for different reasons. Given the wrong notion, one would exclude $\nabla \cdot \vec{E}$ because it is gauge-dependent. Given the right notion (using G), one excludes $\nabla \cdot \vec{E}$ as vanishing. Thus one sees how, in this example, the wrong gauge transformations are consistent with the correct gauge-invariant non-vanishing $\vec{E}$ and $\vec{B}$ parts, the curls.

13 Presupposition of Dirac Observables

The usual concept of "Dirac observables" as entities that Poisson-commute with all first-class constraints is interesting largely on the assumption that a first-class constraint generates a gauge transformation. Now that it is clear that a first-class constraint generally does not generate a gauge transformation, the usual concept of Dirac observable, so defined, is of rather lessened interest, if any. One might nonetheless calculate how the electromagnetic field strength $F_{\mu\nu}$ fares when measured by the crooked rod of Dirac observables as traditionally defined. One can take its Poisson bracket directly once one defines, with Anderson and Bergmann, the Poisson bracket of the time derivative of a canonical coordinate to be the time derivative of the Poisson bracket of the canonical coordinate [1]:

$$\{\dot{q}^A, \} = \frac{\partial \{q^A, \}}{\partial t}. \tag{19}$$

This definition facilitates recapitulating a calculation made above (Eqn. 3) by more pedestrian means. Smearing $p^0(y)$ with arbitrary $\xi(t, y)$ and taking the Poisson bracket gives

$$\delta F_{\mu\nu} = \{F_{\mu\nu}(t,x), \int d^3y p^0(y)\xi(t,y)\} = \{\partial_\mu A_\nu - \partial_\nu A_\mu, \int d^3y p^0(y)\xi(t,y)\} = \partial_\mu \xi \delta^0_\nu - \partial_\nu \xi \delta^0_\mu. \tag{20}$$

Let $\mu = 0$, $\nu = n$:

$$\delta F_{0n} = -\delta \vec{E} = \partial_0 \delta A_n - \partial_n \delta A_0 = \partial_0 \xi \delta^0_n - \partial_n \xi \delta^0_0 = -\partial_n \xi \neq 0. \tag{21}$$

As was also found above, while $\vec{B}$ is unchanged, $\vec{E}$ is changed by $\partial_n \xi$. Hence the electric field is not a Dirac observable by the usual reckoning, which is odd. That is contrary to what Matschull found [28], likely because the temptation to default to the conventional wisdom overwhelmed the motivation to do trivial calculations.

What does the secondary $p^i,_i(x)$ do? That calculation also can be redone using the Poisson bracket now:

$$\delta F_{\mu\nu} = \{F_{\mu\nu}(t,x), \int d^3y p^i,_i(y)\epsilon(t,y)\} = \{\partial_\mu A_\nu - \partial_\nu A_\mu, \int d^3y - p^i(y)\frac{\partial \epsilon}{\partial y^i}\}$$
$$= \partial_\mu \int d^3y \delta(x,y)(-\delta^i_\nu \frac{\partial \epsilon}{\partial y^i}) - \mu \leftrightarrow \nu = \delta^i_\mu \partial_\nu \partial_i \epsilon - \delta^i_\nu \partial_\mu \partial_i \epsilon. \tag{22}$$

Clearly $\vec{B}$ is unchanged, but $\vec{E}$'s change is obtained by setting $\mu = 0$, $\nu = n$:

$$\delta F_{0n} = -\delta \vec{E} = \delta^i_0 \partial_n \partial_i \epsilon - \delta^i_n \partial_0 \partial_i \epsilon = -\partial_n \partial_0 \epsilon. \tag{23}$$

Again $\vec{E}$ is changed by an arbitrary gradient. This is again contrary to Matschull's claim [28].

By now the remedy is clear: the primary and secondary constraints should be suitably combined in G. A plausible replacement for the usual concept of Dirac observable, at least for electromagnetism and other theories with internal symmetries, is to look for quantities that have vanishing Poisson bracket with the gauge generator G. That should suffice for electromagnetism; the field strength $F_{\mu\nu}$ is thus observable. One has

$$\{F_{\mu\nu}(t,x), G\} = \{\frac{\partial}{\partial x^{\mu}}A_{\nu}(t,x) - \frac{\partial}{\partial x^{\nu}}A_{\mu}(t,x), \int d^3y - p^{\sigma}(y)\frac{\partial \epsilon}{\partial y^{\sigma}}\} =$$

$$\int d^3y(-\frac{\partial}{\partial x^{\mu}}\delta^{\sigma}_{\nu}\partial_{\sigma}\epsilon(t,y) + \frac{\partial}{\partial x^{\nu}}\delta^{\sigma}_{\mu}\partial_{\sigma}\epsilon(t,y) = -\partial_{\mu}\partial_{\nu}\epsilon(t,x) + \partial_{\nu}\partial_{\mu}\epsilon(t,x) \equiv 0. \quad (24)$$

Thus the electric and magnetic fields are observable by the appropriate criterion, which uses the gauge generator G rather than any first-class constraint in isolation. For Yang-Mills fields, matters should be more complicated, but still equivalent to the Lagrangian result (where $F^{i}_{\mu\nu}$ is gauge-dependent and hence not observable [35]).

14 Conclusion

Carefully doing Hamiltonian calculations for electromagnetism, as an end in itself, would be using a sledgehammer to crack a peanut. But the pattern of ensuring that the Hamiltonian formalism matches the Lagrangian one, which is perspicuous and correct, will prove very illuminating for the analogous treatment of GR. There the right answers are generally not evident by inspection, and the calculations are difficult and error-prone. Knowing what a properly dotted "i" and a properly crossed "t" look like will be crucial in GR, where various attractive entrenched errors related to the first-class-constraint-generates-a-gauge-transformation theme need to be diagnosed. In particular, one should use the total Hamiltonian and its associated gauge generator G, not the extended Hamiltonian and each first-class constraint smeared separately. While various people have made such advocacy before, it would seem that the calculation of the gauge dependence of the electric field and the spoilage of Gauss's law achieve a new level of rational compulsion for the Lagrangian-equivalent total Hamiltonian and G.

One example of an entrenched error in canonical GR is the common claim that $\mathcal{H}_i$ generates a spatial coordinate transformation. While of course $\mathcal{H}_i$ does have the appropriate Poisson brackets with the spatial metric and its conjugate momentum to generate a spatial coordinate transformation as far as those fields are concerned [35], the falsehood of the statement in classical GR is evident from the Poisson bracket with the shift vector β^j and the lapse function N. The immediate results

$$\begin{aligned}\{\mathcal{H}_i(x), \beta^j(y)\} &= 0, \\ \{\mathcal{H}_i(x), N(y)\} &= 0\end{aligned} \tag{25}$$

do not give even the Lie derivative of a scalar like the lapse N, much less that of a vector like β^i. One can treat $\mathcal{H}_i$ as generating a coordinate transformation on a single initial data surface (much as one can keep $\vec{E}$ from changing due to $p^i{}_{,i}$ if one uses only a time-independent smearing function). But failure to transform the lapse and shift destroys the information that allows the aspiring initial data surface to be embedded consistently into space-time; the aspiring initial data surface instead is just a lonely moment. To recover the usual electromagnetic gauge transformations and GR coordinate transformations, one instead needs the gauge generator to pick out gauge transformations in the Hamiltonian context [20]; G transforms the scalar potential (or lapse and shift) appropriately as well. Taking seriously the gauge generator G, not first class constraints in isolation, as generating gauge transformations will remove the still common expectation [10] that observables should have vanishing Poisson brackets with first class constraints. There might be some clarification achieved for canonical quantization.

15 Appendix: Application to GR

As in electromagnetism, taking a first-class constraint as (by itself) generating a gauge transformation leads to trouble in GR. The momentum constraint in the Lagrangian context is $D^i K_{ij} - D_j K = 0$, the time-space part of Einstein's equations, where the extrinsic curvature tensor K_{ij} tells how space bends relative to space-time. K_{ij} is defined in terms of the spatial metric g_{ij} and the lapse N and shift β^i and some time and space derivatives thereof: $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - D_i\beta_j - D_j\beta_i)$. The lapse N relates physical time to coordinate time, while the shift vector β^i tells how spatial coordinates move over time. The time-time part of Einstein's equations (without sources) is $K_{ij}K^{ij} - (K^i_i)^2 - R = 0$. The primary constraints $p_0 =_{df} \frac{\partial \mathcal{L}}{\partial N_{,0}} = 0$ and $p_i =_{df} \frac{\partial \mathcal{L}}{\partial N_{,i}} = 0$, after the transition to the Hamiltonian formalism, are first-class.

p_0 varies N; p_i varies β^i. Do they generate gauge transformations? Varying N arbitrarily, or β^i, or both, typically *spoils the constraints in Lagrangian form*, just as in electromagnetism. For example, p_i varies β^i, which in the momentum constraint $D^i K_{ij} - D_j K = 0$ introduces new terms

$$(-D^i \frac{1}{2N})(D_i \delta\beta_j + D_j \delta\beta_i) - \frac{1}{2N}(D^i D_i \delta\beta_j + D^i D_j \delta\beta_i) + (D_j N^{-1}) D^i \delta\beta_i + N^{-1} D_j D^i \delta\beta_i,$$

which typically fail to vanish. (Again one notices a Laplacian-type piece.) Likewise with varying N in the momentum constraint or N or β^i in the Hamiltonian constraint. Thus the Gauss-Codazzi relations embedding space into space-time fail if one mistakes first-class constraints p_i or p_0 for generators of gauge transformations.

The constraints $\mathcal{H}_0$ and $\mathcal{H}_i$ in terms of π^{ij} don't notice trouble—they don't even see p_0 or p_i because they are independent of N and β^i. But that fact simply shows that the constraints in Hamiltonian $(q - p)$ form cease to be equivalent to the Lagrangian constraints ($q - \dot{q}$ form). The constraints in Lagrangian form, in terms of K_{ij} rather than π^{ij}, are those with direct physical significance. This error would be obvious if it were common to move from the verbal formula "a first-class constraint generates a gauge transformation" to mathematics; but in fact only the move from mathematics to the verbal formula is generally made. Practical people (like numerical relativists) have no problem, because they'd likely not use first-class constraint transformations instead of coordinate transformations. Thus conceptual confusion is generated without immediate mathematical or empirical difficulty.

15.1 What Do $\mathcal{H}_i$ and $\mathcal{H}_0$ Generate?

One often reads that $\mathcal{H}_i$ generates spatial coordinate transformations. Given the electromagnetic precedent above, one is prepared to disbelieve that claim. Given the distinction between H and G [20], it is clear that $\mathcal{H}_0$ can help either in H to generate time evolution or in G to generate a change of time coordinate. Does $\mathcal{H}_0$ generate some combination of time evolution and change of time coordinate? Apparently not; $\mathcal{H}_0$ has a well defined mathematical action with no obvious interesting physical meaning in isolation. One can find the relation between $\mathcal{H}_0$ and space-time coordinate transformations by starting with the gauge generator G and throwing away some terms to isolate $\mathcal{H}_0$. The gauge generator G has a bunch of terms involving the primary constraints, the lapse and shift, and (in some cases) the spatial 3-metric [20]; these will not affect the 3-metric g_{ij}. Thus $\{g_{ij}(x), G\} = \{g_{ij}(x), \int d^3y[\epsilon^{\perp}(y)\mathcal{H}_0(y) + \epsilon^i(y)\mathcal{H}_i(y)]\}$. $\epsilon^{\perp}$ is the normal projection of the 4-vector ξ^{μ} describing an infinitesimal coordinate transformation, while ϵ^i is the spatial projection. Thus one has $\epsilon^{\perp} = N\xi^0$ and $\epsilon^i = \xi^i + \beta^i \xi^0$.

Setting $\epsilon^i = 0$ to make a purely normal coordinate transformation, one has

$$\{g_{ij}(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\} = \delta^{\mu}_{i}\delta^{\nu}_{j}\pounds_{(\epsilon^{\perp}n^{\alpha})}g_{\mu\nu}(x).$$

While that looks like $\frac{6}{10}$ of the desired space-time Lie derivative formula, the obvious results $\{N(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\} = 0$ and $\{\beta^i(x), \int d^3y \epsilon^{\perp}(y)\mathcal{H}_0(y)\} = 0$ show that the rest of the Lie derivative formula is violated. Thus $\mathcal{H}_0$ does not generate a coordinate transformation.

Likewise $\mathcal{H}_i$ can generate part of a spatial coordinate transformation or part of a spatial translation. One can readily see from the Poisson brackets that $\mathcal{H}_i$ does not generate a coordinate transformation. While the Poisson brackets

$$\begin{aligned} \{g_{ij}(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} &= \pounds_{\xi} g_{ij}(x), \\ \{\pi^{ij}(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} &= \pounds_{\xi} \pi^{ij}(x) \end{aligned} \tag{26}$$

are appropriate for a coordinate transformation, the brackets

$$\begin{aligned} \{\beta^i(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} &= 0, \\ \{N(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} &= 0 \end{aligned} \tag{27}$$

are not appropriate for a coordinate transformation—not even a spatial coordinate transformation. They aren't appropriate for a spatial translation, either. Neither does it seem to be possible to regard the transformation as a combination of a coordinate transformation and a translation. By itself $\mathcal{H}_i$ simply generates variations in the spatial metric g_{ij}, its conjugate momentum π^{ij}, and functionals thereof, variations with a spatial Lie derivative form. The world is thereby changed, but not in a way with any special physical meaning. In fact there seems to be no sensible physical meaning for the transformation in isolation; by itself it is simply a bad change, in that if one starts with a physically allowed situation, a change is made to an impermissible one.

It long was easy to neglect 4-dimensional coordinate transformations because a usable gauge generator was unavailable after the $3+1$ innovation in 1958 [23, 24] rendered the original G [1] obsolete. The $3+1$ G finally appeared in 1982 [20]. The fact that GR lacks hidden symmetries [59] implies that each first-class constraint cannot generate a gauge transformation. There being only 4 coordinate transformations and no other symmetries, the 8 first-class constraints can contribute only in combination(s).

If a first-class constraint generated a gauge transformation, then the gauge generator G would be an *arbitrary* sum of first-class constraints, not a carefully combined sum with twice as many constraints as arbitrary functions (as it in fact is).

It being demonstrated that the first-class secondary constraints $\mathcal{H}_0$ and $\mathcal{H}_i$ do not generate gauge transformations, the question arises what they *do* generate. Given the electromagnetic precedent of the violation of Gauss's law above, one expects that they spoil the Lagrangian constraints, $\frac{4}{10}$ of Einstein's equations, the Gauss-Codazzi relations describing how space fits into space-time. Let us calculate to find out.

For starters, one can see what the primary constraints do. This is easy, because they only change the lapse and shift, which appear with no time derivatives. Thus one not does not have to figure out what to do with velocities in the extrinsic curvature tensor K_{ij}. For p (conjugate to the lapse N), one has

$$\{\int d^3y \epsilon(y) p(y), D_i(K^i_j - \delta^i_j K)(x)\} = D_i[(K^i_j - \delta^i_j K)\epsilon N^{-1}] \neq 0, \qquad (28)$$

even if one uses $D_i(K^i_j - \delta^i_j K) = 0$. So here is one primary first-class constraint that spoils a Lagrangian constraint and thus makes a bad physical change.

What does p do to the $q - \dot{q}$ Hamiltonian constraint, the normal-normal part of Einstein's equations? To answer that question, it is convenient to define a lapse-less factor in the extrinsic curvature tensor: $L_{ij} =_{df} NK_{ij} = \frac{1}{2}(\dot{g}_{ij} - D_i\beta_j - D_j\beta_i)$. Thus

$$\{\int d^3y \epsilon(y) p(y), K^{ij}K_{ij} - K^i_i K^j_j - R(x)\} = \int d^3y \epsilon(y) \{p(y), N^{-2}(x)(L^{ij}L_{ij} - L^2(x))\}$$
$$= 2\epsilon(x) N^{-1}(K^{ij}K_{ij} - K^2) \neq 0. \qquad (29)$$

Thus p spoils all 4 of the constraints in Einstein's equations. That is no surprise: changing the lapse arbitrarily while not changing the shift vector or the spatial metric has no chance of being a coordinate transformation, the only symmetry that Einstein's equations have.

What does p_i do to the $q - \dot{q}$ momentum constraint?

$$\{\int d^3y \epsilon^i(y) p_i(y), D_l(K^l_j - \delta^l_j)(x)\} = D_i(\frac{1}{2N} D_j \epsilon^i) + D^i(\frac{1}{2N} D_i \epsilon_j) - D_j(N^{-1} D_i \epsilon^i) \neq 0.$$

Thus p_i spoils some Einstein equations also—not a surprise from so blunt a tool, which changes the shift vector arbitrarily while leaving everything else alone.

Finally, what does p_i do to the $q - \dot{q}$ Hamiltonian constraint?

$$\{\int d^3y \epsilon^l(y) p_l(y), K^{ij}K_{ij} - K^2 - R(x)\} = (D_j \epsilon^i) \frac{2}{N}(K^j_i - \delta^j_i K)(x) \neq 0.$$

Yet again, a first-class primary constraint spoils a Lagrangian constraint, rendering Einstein's equations false (assuming that one still regards the altered N as the lapse).

For electromagnetism and GR, the fraction of primary constraints that generate gauge transformations is $\frac{0}{5}$, whereas the fraction that generates a bad physical change, violating the field equations, is $\frac{5}{5}$. This isn't a good record for the conventional wisdom.

What do the secondary constraints $\mathcal{H}_i$ and $\mathcal{H}_0$ generate? Given the conventional wisdom and the 'proofs' of the Dirac conjecture, one might expect them to generate gauge transformations; but that claim has been falsified above for electromagnetism by direct calculation. Given the example of electromagnetism above, one expects that $\mathcal{H}_i$ and $\mathcal{H}_0$ also spoil the Lagrangian constraints in Einstein's equations. Qualitatively speaking, this is because $\mathcal{H}_i$ and $\mathcal{H}_0$ change the 3-metric while leaving the lapse and shift alone, a transformation that isn't a coordinate transformation, the only available symmetry.

One momentary difficulty is what to do with the 3-metric's velocities in the extrinsic curvature tensor K_{ij}. The answer is that the changes in the velocities arise from time-differentiating the 3-metric, the change of which comes from the Poisson bracket. Above I simply used the Poisson bracket to find in electromagnetism what $p^i{}_{,i}$ did to A_μ, and then calculated $F_{\mu\nu}$ by differentiation of A_μ. Such direct calculation requires no "definitions" (apart from the uncontroversial definition of the electromagnetic field strength) or "insights", the reliance on which, in place of testing on well-understood examples, too often has generated confusion in constrained dynamics. Thus to find what happens to the extrinsic curvature tensor, one only has to find what happens to the 3-metric (mildly nontrivial), the lapse (nothing), and the shift (nothing), and then use the definition of K_{ij}. Nothing can go wrong (except for performance errors in doing the calculation, which can be corrected by further calculations). By contrast, definitions and insights are not reliably self-correcting and hence are methodologically inferior to calculations for understanding first-class constraints.

It is straightforward to infer the change in K_{ij} from the variation of the 3-metric due to $\mathcal{H}_i$, though one has to take both time and space derivatives. One gets

$$\delta K_{ij} = \frac{1}{2N}\frac{\partial}{\partial t}\pounds_\epsilon g_{ij} - \frac{1}{2N}[(\pounds_\epsilon g_{lj})D_i\beta^l + (\pounds_\epsilon g_{il})D_j\beta^l + \beta^m D_m \pounds_\epsilon g_{ij}].$$

This expression can also be written as

$$\delta K_{ij} = \frac{1}{2N}\frac{\partial}{\partial t}\pounds_\epsilon g_{ij} - \frac{1}{2N}[g_{lj}\beta^m \pounds_\epsilon \Gamma^l_{im} + g_{il}\beta^m \pounds_\epsilon \Gamma^l_{jm} + (D_i\beta^l)\pounds_\epsilon g_{lj} + (D_j\beta^l)\pounds_\epsilon g_{li}].$$

By the same procedure, one finds the variation $\delta D_i(K^i_j - \delta^i_j K)$, taking care to find dependence on the 3-metric in the connection from D_i, the index raising of K_{ij} to

K^i_j, and the index lowering of β^i to β_i. There are enough terms that resemble a spatial coordinate transformation that one can split the transformation into a coordinate transformation and a correction term, the reduced change $\not\delta K_{ij} = \delta K_{ij} - \pounds_\epsilon K_{ij}$. The change comes out to be $\delta D_i(K^i_j - \delta^i_j K) = \pounds_\epsilon D_i(K^i_j - \delta^i_j K) + D_i(h^{il}\not\delta K_{lj} - \delta^i_j h^{ab}\not\delta K_{ab})$. While one would not find the Lie derivative term worrisome, the terms involving $\not\delta K_{ij}$ threaten to spoil the $q - \dot{q}$ momentum constraint.

While the exact expression is a bit complicated, one can learn much about it and compare to electromagnetism by taking a special case, the 0th order approximation, namely, flat space-time in Cartesian coordinates: $g_{\mu\nu} = diag(-1,1,1,1)$, $N = 1$, $\beta^i = 0$, $g_{ij} = \delta_{ij}$. Then one has

$$\delta D_i(K^i_j - \delta^i_j K) = \pounds_\epsilon 0 + \partial_i(\delta^{il}\not\delta K_{lj} - \delta^i_j \delta^{ab}\not\delta K_{ab}).$$

The reduced change in the extrinsic curvature tensor, in turn, is

$$\not\delta K_{ij} = \frac{1}{2}(\dot{\epsilon}_i,_j + \dot{\epsilon}_j,_i).$$

Thus the variation in the $q - \dot{q}$ momentum constraint is

$$\frac{1}{2}(\partial_i \partial_i \dot{\epsilon}_j - \partial_j \partial_i \dot{\epsilon}_i \neq 0).$$

Once again a first-class constraint generates a bad physical change, spoiling part of Einstein's equations. Both the expression and the violation of a physically significant physical law are analogous to the electromagnetic expressions above.

Thus far I have derived the variation in the $q - \dot{q}$ momentum constraint simply using the variation in the 3-metric due to $\mathcal{H}_i$ and the definitions of the extrinsic curvature tensor and the $q - \dot{q}$ momentum constraint; there has been no talk of a Poisson bracket involving velocities. But given that changes in the derivatives of the 3-metric induced by Poisson bracket with $\mathcal{H}_i$ are, well, induced by Poisson bracket with $\mathcal{H}_i$, it is reasonable to stipulate that $\{\frac{\partial g_{ij}}{\partial t}, G\} =_{df} \frac{\partial}{\partial t}\{g_{ij}(x), G\}$ for the gauge generator of spatial coordinate transformations. This is an instance of Anderson and Bergmann's expression (7.6), though theirs is intended more generally. Given the connection between G and Lie differentiation, this feature, which resembles the commutativity of Lie and partial differentiation [60], is welcome. There is now no difficulty in understanding expressions such as $\{\int d^3y D_l(K^l_j - \delta^l_j K)(x), \epsilon^i(y)\mathcal{H}_i(y)\}$. One also gets by Poisson bracket $\{K_{ij}(x), \int d^3y \epsilon^k(y)\mathcal{H}_k(y)\} = \delta K_{ij}$ as given above. Thus one has

$$\{\int d^3y D_l(K^l_j - \delta^l_j K)(x), \epsilon^i(y)\mathcal{H}_i(y)\} = \pounds_\epsilon D_i(K^i_j - \delta^i_j K) + D_i(h^{il}\not\delta K_{lj} - \delta^i_j h^{ab}\not\delta K_{ab}). \quad (30)$$

One is now in a position to see what the spatial gauge generator $G[\epsilon^i, \dot{\epsilon}^i]$ does to the $q-\dot{q}$ momentum constraint $D_i(K^i_j - \delta^i_j K)(x)$. While the general result is somewhat complicated at least *prima facie*, a 0th order approximation is illuminating. Assuming that one has started with flat space-time in Cartesian coordinates, the simplest case ($N = 1$, $\beta^i = 0$, $g_{ij} = \delta_{ij}$, and their derivatives vanish), it follows that only p_i and $\mathcal{H}_i$ make surviving contributions. It is immediately evident that, in this simplest of cases, those two contributions *do in fact cancel*—apart from the Lie derivative term (which itself vanishes in this case because one is starting in flat space-time in Cartesian coordinates and taking the Lie derivative of a vanishing 3-vector).

The difference between the GR and electromagnetic cases involving a leftover Lie derivative term indicates that, at the level of components, one should seek only *covariance* for external symmetries, whereas one has *invariance* for internal gauge symmetries. That is good enough; one can express external coordinate transformations by pointing at the world, but one cannot express internal gauge transformations at all, except verbally/mathematically. Clearly observable features of the world must be *invariant* under our merely verbal/mathematical redescriptions. This distinction (covariance for external symmetries, invariance for internal symmetries) is relevant to properly sorting out Bergmann's concept of observables. Bergmann imported his criterion of vanishing Poisson brackets of observables with symmetry generators from electromagnetism to GR simply by analogy, without reflection on the different types of symmetries involved. One would expect $\{\int d^3y D_l(K^l_j - \delta^l_j K)(x), G[\epsilon^i, \dot{\epsilon}^i]\}$ to vanish exactly (beyond 0th order, where it has been shown already that everything properly), apart from a spatial coordinate transformation, a Lie derivative term.

There are several remaining Poisson brackets between the secondary constraints and the Lagrangian constraints: $\{\int d^3y \epsilon(y)\mathcal{H}_0(y), D_l(K^l_j - \delta^l_j K)(x)\}$ (which has been completed), $\{\int d^3y \epsilon^k(y)\mathcal{H}_k(y), K_{ij}K^{ij} - K^2 - R(x)\}$ (which has been completed and cross-checked and is given below), and $\{\int d^3y \epsilon \mathcal{H}_0(y), K_{ij}K^{ij} - K^2 - R(x)\}$ (which has been completed). Given what has appeared for electromagnetism and what has been found so far for GR, one predicts these Poisson brackets will all be nonzero: the secondary first-class constraints will spoil the physically relevant $q - \dot{q}$ constraints, making 40% of Einstein's equations false if they were true initially. Hence the secondary first-class constraints will all generate bad physical changes, not gauge transformations. The long expressions (not shown here) tend to bear out that expectation.

One now has all the Poisson brackets needed to calculate $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$ (which makes a spatial coordinate transformation and so should leave just a Lie derivative term), $\{G[\epsilon^{\perp}, \dot{\epsilon}^{\perp}], D_l(K^l_j - \delta^l_j K)(x)\}$ (which gives a piece of a time coordinate transformation using the equations of motion), and $\{G[\epsilon^{\perp}, \dot{\epsilon}^{\perp}], K_{ij}K^{ij} - K^2 - R(x)\}$ (which gives a piece of a time coordinate transformation using the equations of motion). The calculation $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$ has been explicitly carried out exactly and indeed leaves just the expected spatial Lie derivative term. To carry it out, one needs the Poisson bracket

$$\{\int d^3y \epsilon^k \mathcal{H}_k(y), K_{ij}K^{ij} - K^2 - R(x)\} = -\pounds_{\epsilon}(K^{ab}K_{ab} - K^2 - R) - \frac{K^{ij} - h^{ij}K}{N}\pounds_{\dot{\epsilon}} h_{ij}$$
$$-2\frac{K^{ij}K_{ij} - K^2}{N}\pounds_{\epsilon} N - 2\frac{K^{ij} - h^{ij}K}{N}\pounds_{\epsilon} D_i\beta_j + \frac{2}{N}(K^{ij} - h^{ij}K)(D_i\beta^l)\pounds_{\epsilon} h_{jl}$$
$$+\frac{2}{N}(K^i_l - K\delta^i_l)\beta^m \pounds_{\epsilon}\Gamma^l_{im}. \quad (31)$$

This expression certainly does not look like 0 or even a spatial Lie derivative; once again a (secondary) first-class constraint makes a bad physical change in isolation, not a gauge transformation.

The calculation of $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$ is moderately long and also interesting. It involves Poisson brackets with a suitably tuned and smeared sum of $\mathcal{H}_i$, p_i and even p. The gauge generator for spatial coordinate transformations, dropping a divergence for conceptual clarity, is [20]

$$G[\epsilon^k, \dot{\epsilon}^k] = \int d^3y[\epsilon^k(y)\mathcal{H}_k + (\pounds_{\epsilon}\beta^k + \dot{\epsilon}^k)p_k + N_{,k}\,\epsilon^k p]. \quad (32)$$

Some highlights of the calculation include a contribution from the Poisson bracket of $\dot{h}_{ab}$ in the extrinsic curvature tensor K_{ab}, the Lie derivative of the Christoffel symbols, cancellation of $\dot{\epsilon}^i$ terms generated by different constraints, cancellation of all the many terms of the form $(D\vec{\beta})(D\vec{\epsilon})$ (where D is the spatial covariant derivative), cancellation of all but the antisymmetric parts of the second covariant derivatives in terms of the form $\beta D^2\epsilon$ and in terms of the form $\epsilon D^2\beta$, and cancellation of the two resulting spatial Riemann tensor terms. (If one thought that the Poisson bracket of a velocity could not be evaluated, but this ambiguity is harmless because the result is always multiplied by 0 [61], one would be stumped. If one thought that such a Poisson bracket were merely 0, then one would not get the correct answer.) Thus one gets the predicted result

$$\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\} = -\pounds_{\epsilon}(K_{ij}K^{ij} - K^2 - R(x)). \quad (33)$$

Roughly speaking, $\mathcal{H}_i$ makes what looks almost like a coordinate transformation on the the 3-metric h_{ij} and its canonical momentum—hence the attraction of the widespread belief that $\mathcal{H}_i$ itself generates a spatial coordinate transformation—while failing to change the lapse and shift. The primary constraint terms in G fill in the gaps.

One potential cause of mistakes is a peculiarity of the Anderson-Bergmann Poisson bracket of a velocity. They say that such a quantity occasionally is necessary [1], though frequently it is multiplied by 0 and so in many cases is not needed. One finds that their formula $\{\dot{y}_A, F\} = \frac{\partial F}{\partial t}$, which is desirable to make gauge transformations and time differentiation commute as they do in the Lagrangian formalism, has the quirk that $\{\dot{y}_A, EF\} \neq E\{\dot{y}_A, F\} + \{\dot{y}_A, E\}F = E\frac{\partial}{\partial t}\{y_A, F\} + F\frac{\partial}{\partial t}\{y_A, E\}$, which tends to vanish because basic Poisson brackets tend to be 0 or '1.' Instead one has $\{\dot{y}_A, EF\} = \frac{\partial}{\partial t}\{y_A, EF\} = \frac{\partial}{\partial t}E(\{y_A, F\} + F\{y_A, E\}) = E\frac{\partial}{\partial t}\{y_A, F\} + \{y_A, F\}\frac{\partial E}{\partial t} + F\frac{\partial}{\partial t}\{y_A, E\} + \frac{\partial F}{\partial t}\{y_A, E\}$. When faced with a Poisson bracket of a velocity, one should evaluate it sooner rather than later. A systematic justification for such results would be desirable, and might be available in a histories-based formalism if not elsewhere. One notices that the usual effort to formulate GR in phase space seems more or less doomed from the start, simply because phase space is modeled on *space*, not spacetime. Theories with velocity-dependent gauge transformations (including GR) and relative simultaneity are more naturally written in, *e.g.*, phase space cross time. The usual idea of "reduced phase space" is also problematic, in that simultaneity-changing changes of time coordinate cannot be divided out of phase space because they were never present in phase space originally. One has to evolve (perhaps forward in some places and backward in others) to get to a different simultaneity hypersurface.

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