Reduction and Causal Set Theory's *Hauptvermutung*

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Reduction and Quantum Gravity

- A successful theory of quantum gravity ought to account for the success of the theories it supplants.
- Philosophically, the role of intertheoretic reduction in this context has received little attention.
- No program has yet accomplished this reductive goal, so analysis must be tentative.

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Theses about the *Hauptvermutung*:

1. It describes the nature of causal set theory's (hoped for) reduction to general relativity.
2. It has yet no precise statement.
   - Cf. Earman[1] on cosmic censorship.
3. Further attention to observables may help clarify it.
4. If a precise version is true, it would be an instance of non-Nagelian reduction.

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- Setting the Stage
    - Motivation: Malament (1977)
    - Causal Set Kinematics
    - The Heuristic *Hauptvermutung*
- The Problems Enter
    - Uniform Embedding
    - Coarse-Graining
    - Spacetime Similarity
        - 3 Approaches
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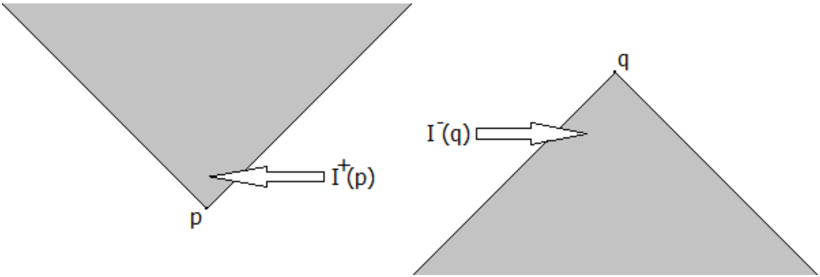
Chronological Future and Past



If $r \in I^-(q)$ one writes $r \ll q$.

Distinguishing Spacetimes

- A spacetime (M, g_{ab}) is *future-distinguishing* when for every $p, q \in M$, $I^+(p) = I^+(q)$ implies $p = q$.
- A spacetime (M, g_{ab}) is *past-distinguishing* when for every $p, q \in M$, $I^-(p) = I^-(q)$ implies $p = q$.
- A spacetime is *distinguishing* when it is both future- and past-distinguishing.

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Malament's Theorem

Theorem

Two distinguishing spacetimes, (M, g_{ab}) and (M', g'_{ab}), must be conformally isometric if there a causal isomorphism $f : M \to M'$. (I.e., if f and f^{-1} preserve the relation $\ll$, then there is a diffeomorphism $\psi : M \to M'$ such that $\psi^* g'_{ab} = \Omega^2 g_{ab}$ for some positive scalar field Ω.)[2]

[2] "The class of continuous timelike curves determines the topology of space-time," *Journal of Mathematical Physics* 18 (1977): 1399–1404.

Interpreting Malament's Theorem

- One can reconstruct a distinguishing spacetime up to a conformal factor by its causal relations alone.
- The conformal factor is related to the spacetime metric's volume element.
- If spacetime were composed of discrete (four-)volume chunks, one could determine volume as well by counting.
- Hence Sorkin's slogan: "Order + Number = Geometry"

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Causal Set Kinematics

A *causal set* $\mathcal{C}$ is an ordered pair $(C, \preceq)$, with a set C and a relation $\preceq$ defined on C such that:

1. (*Reflexivity*) for each $a \in C$, $a \preceq a$;
2. (*Antisymmetry*) for all $a, b \in C$, if $a \preceq b$ and $b \preceq a$, then $a = b$;
3. (*Transitivity*) for all $a, b, c \in C$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$; and
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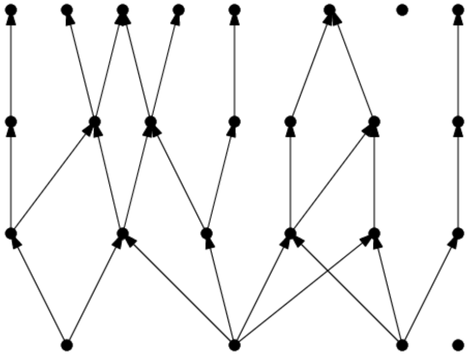
(Almost a) Hasse Diagram



I limit attention to complete causal set histories (or "block universe" causets), which are in some respects "classical" objects.[3]

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[3] Image from p. 230 of Christian Wüthrich, “The Structure of Causal Sets,” *Journal for General Philosophy of Science* 43 (2012): 223–241.

The Intuition Behind the *Hauptvermutung*

- Intuitive idea: if one "samples" enough points with their causal relations from a spacetime, one should be able to recover "most" of the geometry.
- But the causal set is supposed to be more fundamental.
- Thus, inverted: if a causal set is such that it *could have been* "sampled" from a spacetime, then that spacetime is "essentially" unique.

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Faithful Embeddings

Given a causal set $(C, \preceq)$ and a relativistic spacetime (M, g_{ab}), an injection $\phi : C \to M$ is a *faithful embedding* with density ρ when:

1. ϕ preserves causal relations, i.e., for all $a, b \in C$, $a \preceq b$ iff $\phi(a) \ll \phi(b)$; and
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The *Hauptvermutung*

Hauptvermutung If ϕ, ϕ' are faithful embeddings of the causal set $\mathcal{C}$ into relativistic spacetimes $(M, g_{ab}), (M', g'_{ab})$ with density ρ, then (M, g_{ab}) and (M', g'_{ab}) are "approximately isometric above the volume scale ρ^{-1}".

What does it mean for an embedding to be "uniform"?

- Uniformity w.r.t. the volume determined by the metric.
- It is usually argued on grounds of local Lorentz invariance that this must mean uniform on average.

$$P(n, R) = \frac{(\rho V_R)^n e^{-\rho V_R}}{n!}$$

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What does "with high probability" mean?

- Passes a (sufficiently stringent) Fisherian hypothesis test.
- How stringent? Which test statistics?
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Probability and Infinite Structure

- The open problem is how to find a viable battery of tests.
- Typical tests (e.g., χ^2) require finite data, hence finite volume spacetime
- Tests based on partitioning into equal (finite) volume cells not invariant under the choice of partition.
- In general, even extremely unlikely patterns will occur infinitely often in an infinite data set.

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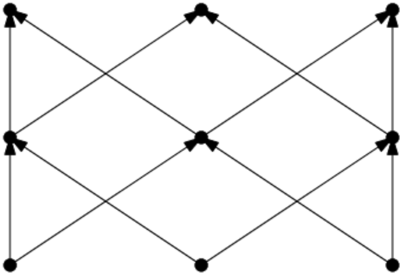
Is requiring an embedding too strong?



It is often remarked that requiring the injection ϕ to be an embedding may be too strong since some small causal sets cannot be embedded.[4]

[4] Image adapted from Christian Wüthrich. "The Structure of Causal Sets," *Journal for General Philosophy of Science* 43 (2012): 223–241.

Coarse-Graining

- Intuitively: for each point, delete with probability p.
- Thus: a coarse-graining is a causet that could have arisen with high probability from a Bernoulli deletion
- Weaken the *hauptvermutung* to embeddings of some coarse-grained causal set
- Same statistical issues arise again for coarse-grainings

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Conceptual Issues

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- Another option: Bernoulli vertex identification on adjacent points (suitably defined)
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Restrictions on Embedding Density

- If the elementary volume is ρ^{-1}, then a coarse-graining with probability *p* yields a new volume $[(1 - p)\rho]^{-1}$
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What is an "approximate isometry"?

- The original proposal involves two "distances" for causal and volumetric structure.
- Measuring differences in volume structure is simple:

$$d_{vol}(g_{ab}, g'_{ab}) = \sup_{p \in M} \left| \ln \frac{\sqrt{|\det(g_{ab})_{|p}|}}{\sqrt{|\det(g'_{ab})_{|p}|}} \right|$$

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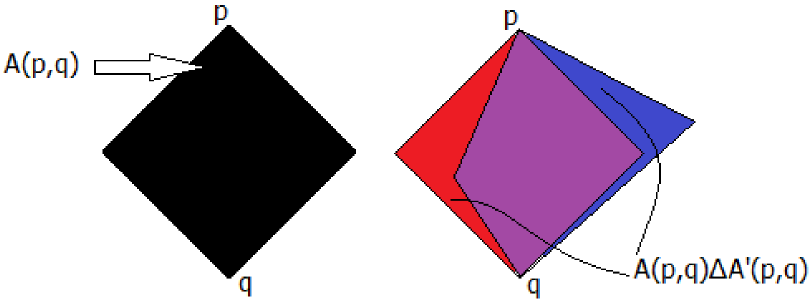
Alexandrov Intervals



Measuring Differences in Causal Structure

$$\alpha(p,q;g_{ab},g'_{ab}) = \begin{cases} \frac{V(A(p,q)\Delta A'(p,q))}{V(A(p,q)\cup A'(p,q))}, & \text{if } 0 < V(A(p,q)\cup A'(p,q)) < \infty \\ 0, & \text{otherwise,} \end{cases}$$

$$d^l_{cau}(g_{ab},g'_{ab}) = \sup_{p,q\in M: V(A(p,q)\cup A'(p,q))\geq l^D} \alpha(p,q;g_{ab},g'_{ab})$$

l is the elementary (Planck) length and l^D the elementary volume.

Problems with the Proposal

- Too fine: compact non-conformal perturbations of Minkowski spacetime are at a maximal distance
- Noldus[5] proposed a correction to solve this problem, but it introduces a host of other problems
- The whole scheme is restricted to spacetimes defined on the same manifold.
    - Causal set theorists want to capture a notion of "scale-dependent" topology.

[5] "A new topology on the space of Lorentzian metrics on a fixed manifold," *Classical and Quantum Gravity* 19 (2002): 6075–6107.

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Bombelli's Statistical Proposal[6]

- Sample n points at random from a spacetime and consider the induced causal structure
- One can write down an expression for $P_n(\mathcal{C}|G)$, the probability that drawing n such points from isometry class G will yield causal set $\mathcal{C}$.
- Idea: compare isometry classes according to $P_n(\mathcal{C}|G)$ for all $|C| = n$, where n is determined by the embedding density
- E.g., since $\sum_{\mathcal{C}} P_n(\mathcal{C}|G) = 1$, $\sqrt{P_n(\mathcal{C}|G)}$ can be interpreted as coordinates on a high-dimensional sphere

[6] "Statistical Lorentzian geometry and the closeness of Lorentzian manifolds," *Journal of Mathematical Physics* 41.10 (2000): 6944–6958.

Bombelli's Statistical Proposal[6]

- Sample n points at random from a spacetime and consider the induced causal structure
- One can write down an expression for $P_n(\mathcal{C}|G)$, the probability that drawing n such points from isometry class G will yield causal set $\mathcal{C}$.
- Idea: compare isometry classes according to $P_n(\mathcal{C}|G)$ for all $|C| = n$, where n is determined by the embedding density
- E.g., since $\sum_{\mathcal{C}} P_n(\mathcal{C}|G) = 1$, $\sqrt{P_n(\mathcal{C}|G)}$ can be interpreted as coordinates on a high-dimensional sphere

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Problems with the Statistical Proposal

- Many inequivalent ways to encode differences in the $P_n(C|G)$ with a distance function
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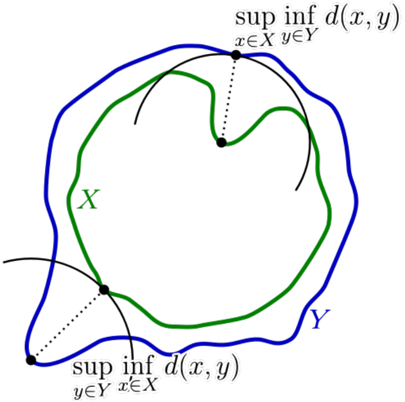
Hausdorff Distance



$$d_H(X,Y) = \max\{\sup_{x\in X}\inf_{y\in Y} d(x,y), \sup_{y\in Y}\inf_{x\in X} d(x,y)\}$$

Gromov-Hausdorff Distance

- The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of $d_H(f[X], f'[Y])$ for all isometric embeddings $f : X \to Z$ and $f' : Y \to Z$ be isometric embeddings
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Lorentz Spaces

- A Lorentz distance is a function $d : M \to [0, \infty]$ satisfying
  1. $d(x, x) = 0$ for all $x \in M$;
  2. if $d(x, y) > 0$, then $d(y, x) = 0$ for all $x, y \in M$; and
  3. if $d(x, y)d(y, z) > 0$, then $d(x, z) \geq d(x, y) + d(y, z)$ for all $x, y, z \in M$.
- Each time-oriented relativistic spacetime (M, g_{ab}) has a Lorentz distance d_g defined by future-directed timelike geodesic distance.

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  1. $|d_{g'}(\psi(p_1), \psi(p_2)) - d_g(p_1, p_2)| < \epsilon$ for all $p_1, p_2 \in M$, and
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- Noldus[7] then proposed

$$d_{GH}((M, g_{ab}), (M', g'_{ab})) = \inf\{\epsilon > 0 : (M, g_{ab}) \approx_\epsilon (M', g'_{ab})\}$$

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- The relationship with observables has not been worked out in much detail.
- One version is restricted to compact globally hyperbolic spacetimes.
- Another (recent![8]) version permits distinguishing spacetimes under some technical conditions whose interpretation is not clear.
- If one can make causal sets into Lorentz spaces, then one could compare causal sets and spacetimes directly without the intermediary of a faithful embedding.

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