The geometry of black hole entropy

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March 13, 2015

The laws of BHT

0. κ is constant on the horizon
1. $\delta M = \frac{1}{8\pi}\kappa\,\delta A + \Omega\,\delta J$
2. $\delta A \geq 0$ in any process
3. $\kappa = 0$ not achievable by any process

The laws of BHT

1. $\delta M = \frac{1}{8\pi}\kappa\,\delta A + \Omega\,\delta J$
2. $\delta A \geq 0$ in any process

Puzzles

1. The δ acting on M and J represents a perturbation of a quantity at spatial infinity; δA is a perturbation at the horizon. How do they relate? (Curiel 2014)
2. How do the δA in the first and second laws relate?

Wald entropy

Iyer and Wald (1994) propose a definition of entropy that will help us answer these. The Wald entropy applies to any diffeomorphism covariant Lagrangian field theory of the form

$$L\left(g_{ab}, \mathring{\nabla}_{a_1} g_{ab}, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_k)} g_{ab}, \psi, \mathring{\nabla}_{a_1} \psi, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_l)} \psi, \mathring{\gamma}\right)$$

On this definition, the perturbations do not act at spatial infinity and the horizon, but globally.

Wald entropy

Two steps

1. Show that

$$L\left(g_{ab}, \mathring{\nabla}_{a_1} g_{ab}, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_k)} g_{ab}, \psi, \mathring{\nabla}_{a_1}\psi, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_l)}\psi, \mathring{\gamma}\right)$$

   may be written~~may be written~~

$$L\left(g_{ab}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{(a_1} \cdots \nabla_{a_m)} R_{bcde}, \psi, \nabla_{a_1}\psi, \ldots, \nabla_{(a_1} \cdots \nabla_{a_l)}\psi\right)$$

   with R_{bcde} the Riemann tensor of g_{ab}.
2. Construct differential forms satisfying the first law using our new L.

Let M be some spacetime, and consider a bundle $\pi : E \to M$ over it.

The 1-jet bundle $\pi^1 : J^1E \to M$ adds data about first derivatives.

The 2-jet bundle $\pi^2 : J^2E \to M$ adds data about second derivatives.

- The (∞-)jet bundle $\pi^\infty : JE \to M$ is the inverse limit $JE = \varprojlim J^nE$.

$$\begin{array}{ccccc} & & JE & & \\ & \swarrow^{\pi^\infty_{n+1}} & & \searrow^{\pi^\infty_{n}} & \\ \cdots \xrightarrow{\pi^{n+2}_{n+1}} & J^{n+1}E & \xrightarrow{\pi^{n+1}_{n}} & J^nE & \xrightarrow{\pi^{n}_{n-1}} \cdots \end{array}$$

- Above some point $p \in M$, JE has all possible Taylor series around p. It gives all the ways a section of E could look over an infinitesimal region.
- For any section ϕ of E, there is a section $j^\infty\phi$ of E that assigns to $p \in M$ the Taylor expansion of ϕ about p.

Variational bicomplex

- The variational bicomplex $\Omega^{*,*}(JE)$ of E is the de Rham complex of differential forms on JE with the exterior derivative $d + \delta$.
- For a Lagrangian L, the global first variational formula

$$\delta L = E + d\Theta$$

  is an equation of $(n, 1)$-forms on JE.
- Wald's locally constructed forms $\Omega^*_{\text{loc}}(M)$ are the image of the pullback along

$$e_\infty : M \times \Gamma(E) \xrightarrow{(\mathrm{id}, j^\infty)} M \times \Gamma(JE) \xrightarrow{\mathrm{ev}} JE$$

Global equation of first variation (Zuckerman, 1987)

Theorem

For any $(n, 0)$-form L, there is an $(n, 1)$-form E and an $(n-1, 1)$-form Θ such that

$$\delta L = E + d\Theta$$

Example: The Gauss–Bonnet theorem (Anderson 1989, xxii–xxv)

The Gauss–Bonnet theorem:

$$\int_M K\, dA = 2\pi\chi(X)$$

for a 2D Riemannian X with Gaussian curvature K.

Take $E = \mathbb{R}^2 \times \mathbb{R}^3$ over $M = \mathbb{R}^2$, restrict attention to local, regularly parametrized surfaces.

$\delta L = d\eta$, so the LHS vanishes.

What gives?

- We're missing the *global* aspects of the problem.
  1. The problem should be invariant under Euclidean motions in $\mathbb{R}^3$ and orientation-preserving diffeomorphisms of the base space.
  2. If we consider only equivariant forms, then we no longer have $\delta L = d\eta$.
  3. This leads us to the *global* first variational formula

$$\delta L = E + d\Theta$$

- Lesson: the global first variational formula $\delta L = E + d\Theta$ incorporates covariance, and encodes global information about the bundle of interest.

Iyer and Wald step 1

- A Lagrangian is determined by a function $L : JE \to \mathbb{R}$.
- IW's result: there is a bijection between functions

$$JE_g \times_M JE_\psi \times_M E_{\overset{\circ}{\gamma}} \to \mathbb{R}$$

and functions

$$JE_g \times_M JE_R \times_M JE_\psi \to \mathbb{R}$$

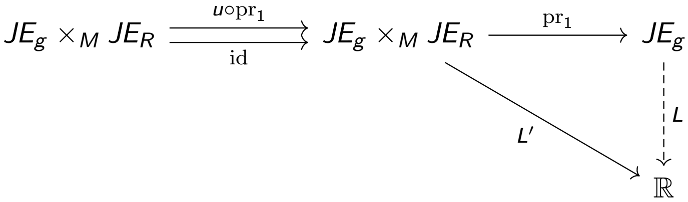
such that L uses the $R^a{}_{bcd}$ of g_{ab}.

Proof

- Consider the function

$$u : J^2 E_g \to E_g \times_M E_R$$
$$u : (p, g_{\mu\nu}, g_{\mu\nu,\lambda}, g_{\mu\nu,\lambda\sigma}) \mapsto ((p, g_{\mu\nu}), (p, R^{\mu}{}_{\nu\lambda\sigma}))$$

- u splits the projection $JE_g \times_M JE_R \to JE_g$ (i.e., $\mathrm{pr}_1 \circ u = \mathrm{id}$), so



is an absolute coequalizer.

Diffeomorphism invariance

We've shown that L can always be rewritten

$$L\left(g_{ab}, \mathring{\nabla}_{a_1} R_{bcde}, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_m)} R_{bcde}, \psi, \mathring{\nabla}_{a_1}\psi, \ldots, \mathring{\nabla}_{(a_1} \cdots \mathring{\nabla}_{a_l)}\psi, \mathring{\gamma}\right)$$

But what about the background fields?
IW claim that they drop out when we demand covariance, but I disagree.

(Assumptions about) general covariance

- General covariance: $\mathrm{Diff}(M)$-equivariance
- Background field: a section of a bundle with a trivial $\mathrm{Diff}(M)$-action
- So by general covariance of L, we must have

$$\frac{\partial L}{\partial \mathring{\gamma}} \mathcal{L}_\xi \mathring{\gamma} = 0$$

  for ξ the infinitesimal generator of the diffeomorphism.
- For a background field, ξ generates id; i.e., $\xi = 0$.

Step 1 summary

1. The variational bicomplex allows for a bit more precision about the geometric objects involved and the role of general covariance.
2. There is a simple bijective correspondence between covariant Lagrangians

$$L : JE_g \times_M JE_\psi \to \mathbb{R}$$

   and covariant Lagrangians

$$L' : JE_g \times_M JE_R \times_M JE_\psi \to \mathbb{R}$$

   which factor through u.
3. Assuming the definition of general covariance just given, we cannot eliminate dependence on background fields.

The first law

- Recall that for any Lagrangian L, there are E and Θ satisfying

$$\delta L = E + d\Theta$$

- The ambiguities in E and Θ are well understood.
- E suffices to pick out the solutions to the equations of motion, but Θ is needed to characterize conserved quantities, like $\delta\Theta$ (Noether, 1918).
- IW define black hole entropy, and derive the first law, by considering a decomposition of Θ.

Θ as a current

- Consider the purely geometric sector: no matter fields, no background fields, and fix the Einstein–Hilbert Lagrangian.
- $\delta\Theta$ is a conserved current, called the Crnković–Witten current, Ashtekar–Bombelli–Koul current, or universal current.
- Pick some spatial slice Σ of M, and define

$$\omega_\Sigma = \int_\Sigma \delta\Theta$$

- On shell, ω_Σ depends only on the homology class of Σ.
- Alternatively: don't integrate, then $\delta\Theta$ defines a cohomology class in $H^{4+1}(M \times \mathcal{S}, \mathbb{R})$, for $\mathcal{S} \subseteq \Gamma(JE_g \times_M JE_R)$ the solution set.

Decomposing Θ

- Now we break the rewriting symmetry. On $JE_g \times_M JE_R$ we have

$$\delta L = E_g\,\delta g + E_R\,\delta R + d\Theta$$

- For a spatial slice Σ, we define

$$S_\Sigma = 2\pi \int_\Sigma E_R\, d\Sigma$$

- If ξ is a stationary Killing field, then it is also a symmetry of L. Noether's theorems give conserved charge Q in terms of $\delta\Theta$ that's closed on shell. Using ξ to decompose Q gives:

$$0 = dQ = -d\delta M + d\left(\frac{1}{4}\kappa\,\delta(E_R\,d\Sigma)\right) + d\left(\Omega\,\delta J\right)$$

- Integrating over Σ and applying Stokes' theorem gives the first law:

$$\delta M = \frac{1}{8\pi}\kappa\,\delta S_\Sigma + \Omega\,\delta J$$

- Takeaway: this derivation of the first law is determined by ξ, the cohomology class of Σ, and E_R.

Puzzle 1

Q: The δ acting on M and J represents a perturbation of a quantity at spatial infinity; δA is a perturbation at the horizon. How do they relate? (Curiel 2014)

A: Neither represent local perturbations, they are covariant perturbations integrated over space at a time. So they are global twice over: once on covariance considerations, once because they are integrals over all of space.

Puzzle 2

Q: How do the δA in the first and second laws relate?

A: A in the second law is the area of the event horizon, obtained by integrating locally over the horizon. The δ there is a finite difference, made non-negative by the local expansion parameter.

A in the first law comes about via Stokes' theorem, as we integrate a global form over a spatial slice. It requires a Killing field for the first law to obtain. It encodes cohomological data about Σ and the Killing field. It requires distinguishing between variations in g_{ab} and variations in $R^a{}_{bcd}$.

New puzzles

- What are the precise notions of locality involved in the answer to puzzle 1? Can we devise local quantities in black hole spacetimes? (Khavkine 2015)
- In drawing a substantial analogy between ordinary thermodynamics and black holes, which version of entropy should we use?

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