CLASSICAL AND QUANTUM OBSERVABLES

In the quantum physics literature, it is generally claimed that the physical content of a quantum theory lies in its algebra of observables. Hence, to identify the observables of a quantum theory is a *conditio sine qua non* to determine what the world must be like if the theory is to be true of the world; one would begin by making clear just what conditions a physical quantity should satisfy to qualify as an observable. However, this basic question is as yet an open problem.

Diana Taschetto

Email: dtaschetto@usp.br

Philosophy Department

University of São Paulo
University of Illinois Chicago

1. Introduction

1.1 Classical Physics Basics

- For a classical physical system with n degrees of freedom:
  - The manifold of points of the cotangent bundle Γ denote the possible states;
  - The regular functions on Γ denote the classical observables;
  - Dynamics manifests itself via a Hamiltonian vector field;
  - The space of classical observables is endowed with the structures of an associative Abelian algebra and of a Lie algebra induced by the Poisson bracket.

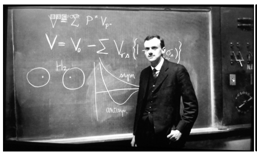
1.2 Quantization: The Canonical Procedure

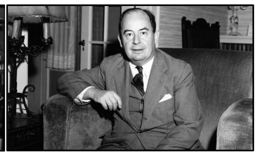
- The prescription (Dirac 1926, 1930, 1957): Replace the Poisson brackets between classical observables by ħ/i times the commutator between the corresponding quantum observables.
- The physical basis: "In a recent paper Heisenberg* puts forward a new theory that suggests that it is not the equations of classical mechanics that are in any way at fault, but that the mathematical operations by which physical results are deduced from them require modification. *All* the information supplied by the classical theory can thus be made use of in the new theory." (Dirac 1926, 642, italics in the original)

*Heisenberg, W. (1925) "Quantum-Theoretical Reinterpretation of Kinematic and Mechanical Relations." *Zs. Phys.* 33, 879-893.

Box 1. Application in Quantum Gravity

"One of the basic difficulties in quantum gravity is to understand which physical quantities should be predicted by the theory. The answer seems simple: the observable quantities in the quantum theory should be the same as in the classical theory, or at least a subset of these. Rather remarkably, however, the problem of what precisely is observable is far from being trivial even in classical general relativity." (Rovelli 1991, 297)





2. The Formation of Quantum Operators Corresponding to Dynamical Variables

2.1 Principles

- The classical coordinate q and momentum p satisfying $(p, q) = 1$ are the generators of the commutative ring of classical quantities $a(q, p)$.
- The quantum operators **p** and **q** satisfying $[\mathbf{p}, \mathbf{q}] = i\hbar$ are the generators of the non-commutative ring of quantum operators **a**.

Rule of Correspondence
(von Neumann 1932)

1. If $a \Leftrightarrow \mathbf{a}$, then $f(a) \Leftrightarrow f(\mathbf{a})$;
2. If $a \Leftrightarrow \mathbf{a}$ and $b \Leftrightarrow \mathbf{b}$, then $a + b \Leftrightarrow \mathbf{a} + \mathbf{b}$.

2.2 Consequences

Consequence I

From 1 and 2, we have

$$(a + b)^2 - a^2 - b^2 = ab + ba \Leftrightarrow \mathbf{ab} + \mathbf{ba} \quad \text{(I)}$$

and so

$$a(ab + ba) + (ab + ba)a - a^2b - b^2a = 2aba \Leftrightarrow \mathbf{2aba}. \quad \text{(II)}$$

Using this,

$$(ab + ba)^2 - b(2aba) - (2aba)b = -(ab - ba)^2 \Leftrightarrow -(\mathbf{ab} - \mathbf{ba})^2 \quad \text{(III)}$$

and hence

$$(ab - ba) \Leftrightarrow \pm(\mathbf{ab} - \mathbf{ba}). \quad \text{(IV)}$$

It will now follow from (IV) and (I) that the rings must be isomorphic:

$$ab \Leftrightarrow \mathbf{ab} \text{ for all } a \text{ and } b \text{ or } ab \Leftrightarrow \mathbf{ba} \text{ for all } a \text{ and } b.$$

Hence, if one of the rings is commutative (like the ring of classical observables) and the other is not (like the ring of quantum observables), 1 and 2 are inconsistent.

Consequence II

Given

$$ab = 1/2(a + b)^2 - 1/2(a - b)^2,$$

one derives the recognized rule for all double products:

$$ab \Leftrightarrow 1/2(\mathbf{ab} + \mathbf{ba}).$$

From this rule,

$$abc \Leftrightarrow 1/4(\mathbf{ab} + \mathbf{ba})\mathbf{c} + 1/4\mathbf{c}(\mathbf{ab} + \mathbf{ba}),$$

$$cab \Leftrightarrow 1/4(\mathbf{ca} + \mathbf{ac})\mathbf{b} + 1/4\mathbf{b}(\mathbf{ca} + \mathbf{ac}),$$

$$bca \Leftrightarrow 1/4(\mathbf{bc} + \mathbf{cb})\mathbf{a} + 1/4\mathbf{a}(\mathbf{bc} + \mathbf{cb}).$$

Hence,

$$\mathbf{a}(\mathbf{bc} - \mathbf{cb}) = (\mathbf{bc} - \mathbf{cb})\mathbf{a};$$

similarly for the two other expressions. This shows that the commutator $(\mathbf{bc} - \mathbf{cb})$ must commute with any other quantum observables. Schur's lemma requires that $(\mathbf{bc} - \mathbf{cb})$ be a multiple $\lambda(\mathbf{bc})I$ of the unit operator I in any irreducible representation. Thus, if $ab \Leftrightarrow \mathbf{x}$, then

$$\lambda(\mathbf{xc})I = \mathbf{xc} - \mathbf{cx} = 1/2(\mathbf{ab} + \mathbf{ba})\mathbf{c} - 1/2\mathbf{c}(\mathbf{ab} + \mathbf{ba}) = \lambda(\mathbf{bc})\mathbf{a} + \lambda(\mathbf{ac})\mathbf{b}.$$

It follows that $\lambda(\mathbf{xc}) = \lambda(\mathbf{bc}) = \lambda(\mathbf{ac}) = 0$ and so all the commutators must vanish. Thus, any quantum observable must commute with any other quantum observable; this fact flatly contradicts the basic postulate of quantum mechanics.

3. Final remarks

- Historically: "There was rarely in the history of physics a comprehensive theory which owed so much to one principle as quantum mechanics owed to Bohr's correspondence principle." (Jammer 1966, 118)
- It follows from 2.2 that the correspondence principle is false.

References

Dirac, P. A. (1926) *Proc. Roy. Soc. A* 109, 642-653.
Dirac, P. A. (1930) *The Principles of Quantum Mechanics*. Oxford: Clarendon Press.
Dirac, P. A. (1957) *Lectures on Quantum Mechanics*. New York: Dover.
Groenewold, H. J. (1946) *Physica* (12) 7, 405-448.
Jammer, M. (1966). The Conceptual Development of Quantum Mechanics. New York: McGraw-Hill.
Rovelli, C. (1991) *Class. Quantum Grav.* (2) 8, 297-328.
Temple, G. (1935) *Nature* 135, 957.

Acknowledgments

FAPESP

Grant number 2020/02969-2.